Schrödinger's Cat

One of the interpretational problems of QM consists in a fact that the system can be in a superposition of two states $|\phi\rangle$ and $|\psi\rangle$given as

$$\sqrt{\frac{1}{2}}\left(|\phi\rangle + |\psi\rangle\right)$$

even if being in one of these states excludes the another one. A typical example is a superposition of two states of a cat being alive or dead. While quantum superposition of microscopic states is not particularly strange, as it is essential for quantum interference effects, a superposition of macroscopic, *classical* states (like a cat) seems to be paradoxical. There is one very important feature that defines a macroscopic state: it is a state that is by itself a superposition of a large number of single microscopic states. We will show that it is possible to construct a superposition of classical antinomic states, however such superpositions are practically not detectable and very fragile.

Harmonic oscillator - reminder

Consider one-dimensional harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{1}$$

that we will solve with the help of creation and annihilation operators. It is convenient to define dimensionless operators

$$\hat{\xi} = \sqrt{\frac{m\omega}{\hbar}}\hat{x},\ \hat{\pi} = \frac{1}{\sqrt{m\hbar\omega}}\hat{p}. \tag{2}$$

Then

$$\hat{H} = \frac{1}{2}\hbar\omega\left(\hat{\pi}^2 + \hat{\xi}^2\right) \tag{3}$$

and

$$\hat{a} = \sqrt{\frac{1}{2}}\left(\hat{\xi} + i\hat{\pi}\right),\ \hat{a}^\dagger = \sqrt{\frac{1}{2}}\left(\hat{\xi} - i\hat{\pi}\right) \tag{4}$$

and

$$\begin{aligned}\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right),\ \hat{\xi} = \sqrt{\frac{1}{2}}\left(\hat{a}^\dagger + \hat{a}\right), \\ \hat{p} &= i\sqrt{\frac{m\omega\hbar}{2}}\left(\hat{a}^\dagger - \hat{a}\right),\ \hat{\pi} = i\sqrt{\frac{1}{2}}\left(\hat{a}^\dagger - \hat{a}\right).\end{aligned} \tag{5}$$

Note that

$$\left[\hat{\xi}, \hat{\pi}\right] = i,\ \left[\hat{a}, \hat{a}^\dagger\right] = 1. \tag{6}$$

Recall that

$$\begin{aligned}
\hat{a}^\dagger \hat{a} \; |n\rangle &= n \, |n\rangle \, , \\
\hat{a} \; |n\rangle &= \sqrt{n} \, |n-1\rangle \, , \\
\hat{a}^\dagger \, |n\rangle &= \sqrt{n+1} \, |n+1\rangle
\end{aligned} \tag{7}$$

and

$$\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} \, + \frac{1}{2} \right) . \tag{8}$$

In configuration representation $\hat{\pi} = -i\partial/\partial\xi$ and in momentum representation $\hat{\xi} = i\partial/\partial\pi$.

Coherent states

A good model for a classical state is a *coherent state*, i.e. normalized eigen-state of the annihilation operator $\hat{a}$:

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0} \frac{z^n}{\sqrt{n!}} \, |n\rangle \tag{9}$$

where z is a complex number. Indeed

$$\begin{aligned}
\hat{a} \; |z\rangle &= e^{-|z|^2/2} \sum_{n=1} \frac{z^n}{\sqrt{n!}} \sqrt{n} \, |n-1\rangle \\
&= e^{-|z|^2/2} \sum_{n=0} \frac{z^{n+1}}{\sqrt{n!}} \sqrt{n} \, |n\rangle \\
&= z \; |z\rangle \, .
\end{aligned} \tag{10}$$

This means

$$\langle z| \, \hat{a}^\dagger = \langle z| \, z^* \tag{11}$$

Let's discuss some properties of coherent states.

Completness:

$$\frac{1}{\pi} \int d^2z \; |z\rangle \, \langle z| = \mathbf{1}. \tag{12}$$

We can prove that using (9):

$$\frac{1}{\pi} \int d^2z \; |z\rangle \, \langle z| = \frac{1}{\pi} \sum_{m,n} \frac{1}{\sqrt{n!m!}} \, |n\rangle \, \langle m| \int d^2z e^{-|z|^2} z^n (z^*)^m . \tag{13}$$

Using $z=\rho e^{i\varphi}$ and $d^2z = d\varphi\,\rho d\rho$ we get:

$$\begin{aligned}\int d^2z e^{-|z|^2} z^n (z^*)^m &= \underbrace{\int_0^\pi d\varphi\, e^{i(n-m)\varphi}}_{2\pi\delta_{nm}} \underbrace{\int_0^\infty d\rho\, e^{-\rho^2}\rho^{n+m+1}}_{\rho^2=t,\ d\rho\rho=1/2\,dt} \\ &= \pi\delta_{nm}\underbrace{\int_0^\infty dt\, e^{-t}t^n}_{n!}, \end{aligned} \tag{14}$$

which gives:

$$\frac{1}{\pi}\int d^2z\ |z\rangle\langle z| = \sum_n |n\rangle\langle n| = \mathbf{1}. \tag{15}$$

"Orthogonality":

$$\begin{aligned}\langle y|\,z\rangle &= e^{-(|y|^2+|z|^2)/2}\sum_{m,n}\frac{(y^*)^m z^n}{\sqrt{n!m!}}\langle m|\,n\rangle = e^{-(|y|^2+|z|^2)/2}\sum_n \frac{1}{n!}(y^* z) \\ &= \exp\left(-\frac{1}{2}(|y|^2+|z|^2)+y^*z.\right) \end{aligned}\tag{16}$$

When $y=z$ this gives 1. It is easy to show that probability

$$|\langle y|\,z\rangle|^2 = e^{-|z-y|^2}. \tag{17}$$

Probability distribution:

$$P_n(z) = |\langle n|\,z\rangle|^2 = \frac{|z|^{2n}\,e^{-|z|^2}}{n!}. \tag{18}$$

Note that mean value of the number operator $N=\hat{a}^\dagger\hat{a}$ is:

$$\bar{n} = \langle z|\,\hat{a}^\dagger\hat{a}\,|z\rangle = |z|^2 \tag{19}$$

hence

$$P_n(z) = \frac{1}{n!}\bar{n}^n e^{-\bar{n}} \tag{20}$$

is a Poisson distribution.
Mean energy:

$$\langle z|\,\hat{H}\,|z\rangle = \hbar\omega\ \langle z|\left(\hat{a}^\dagger\hat{a}\ +\frac{1}{2}\right)|z\rangle = \hbar\omega\ \left(|z|^2+\frac{1}{2}\right). \tag{21}$$

Mean position and momentum:

$$\bar{x} = \langle z|\,\hat{x}\,|z\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,(z^*+z)\,,\ \bar{p} = \langle z|\,\hat{p}\,|z\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\,(z^*-z)\,. \tag{22}$$

Mean square deviations:

$$\Delta x^2 = \langle z| (\hat{x} - \bar{x})^2 |z\rangle = \langle z| \hat{x}^2 - 2\bar{x}\hat{x} + \bar{x}^2 |z\rangle = \langle z| \hat{x}^2 |z\rangle - \bar{x}^2. \quad (23)$$

Note that

$$\begin{aligned} \hat{x}^2 &= \frac{\hbar}{2m\omega} \left(\hat{a}^\dagger \hat{a}^\dagger + \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right) \\ &= \frac{\hbar}{2m\omega} \left(\hat{a}^\dagger \hat{a}^\dagger + 2\hat{a}^\dagger \hat{a} + \hat{a}\hat{a} + 1\right). \end{aligned} \quad (24)$$

Hence

$$\begin{aligned} \Delta x^2 &= \frac{\hbar}{2m\omega} \left[(z^* + z)^2 + 1 - (z^* + z)^2\right] \\ &= \frac{\hbar}{2m\omega}. \end{aligned} \quad (25)$$

Similarly

$$\Delta p^2 = \langle z| \hat{p}^2 |z\rangle - p^2 \quad (26)$$

with

$$\begin{aligned} \hat{p}^2 &= -\frac{m\omega\hbar}{2} \left(\hat{a}^\dagger \hat{a}^\dagger - \hat{a}^\dagger \hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right) \\ &= -\frac{m\omega\hbar}{2} \left(\hat{a}^\dagger \hat{a}^\dagger - 2\hat{a}^\dagger \hat{a} + \hat{a}\hat{a} - 1\right) \end{aligned} \quad (27)$$

and

$$\begin{aligned} \Delta p^2 &= -\frac{m\omega\hbar}{2} \left[(z^* - z)^2 - 1 - (z^* - z)^2\right] \\ &= \frac{m\omega\hbar}{2}. \end{aligned} \quad (28)$$

Note that coherent states for any z saturate uncertainty principle (like the ground state of the harmonic oscillator)

$$\Delta x^2 \Delta p^2 = \frac{\hbar^2}{4}. \quad (29)$$

Wave functions:
To calcuate explicit form of the wave functions we shall use (4):

$$\sqrt{\frac{1}{2}} \left(\xi + \frac{d}{d\xi}\right) \psi_z(\xi) = z\psi_z(\xi). \quad (30)$$

The solution reads:

$$\psi_z(\xi) = C \exp\left(-\frac{1}{2}(\xi - \sqrt{2}z)^2\right). \quad (31)$$

Similarly in the momentum space:

$$i\sqrt{\frac{1}{2}}\left(\pi + \frac{d}{d\pi}\right)\tilde{\psi}_z(\pi) = z\tilde{\psi}_z(\pi). \tag{32}$$

And the solution corresponds to $\psi_z(\xi)$ with $z \to -iz$:

$$\tilde{\psi}_z(\pi) = \tilde{C}\exp\left(-\frac{1}{2}(\pi + i\sqrt{2}z)^2\right). \tag{33}$$

Probability distributions:
Let's compute the probability distribution fot two cases $z = i\rho$ and $z = \rho$, with ρ being real and positive.

- Case: $z = i\rho$

$$\begin{aligned} P_z(\xi) &= |\psi_z(\xi)|^2 = |C|^2 \exp\left(-\frac{1}{2}(\xi + i\sqrt{2}\rho)^2 - \frac{1}{2}(\xi - i\sqrt{2}\rho)^2\right) \\ &= |C|^2 \exp(2\rho^2)\exp\left(-\xi^2\right) \end{aligned} \tag{34}$$

  and

$$P_z(\pi) = \left|\tilde{C}\right|^2 \exp\left(-(\pi - \sqrt{2}\rho)^2\right). \tag{35}$$

  In this case space distribution is proportional to a a Gaussian centred at $\xi = 0$ and is independent of the sign of ρ. On the contrary, momentum distribution is a Gaussian centred at $\pi = \sqrt{2}\rho$ and depends on the sign of ρ. Therefore two states $z = \pm\iota\rho$ correspond to two antinomic states in momentum space.

- Case: $z = \rho$

$$P_z(\xi) = |\psi_z(\xi)|^2 = |C|^2 \exp\left(-(\xi + \sqrt{2}\rho)^2\right) \tag{36}$$

  and

$$\begin{aligned} P_\pi(\xi) &= \left|\tilde{\psi}_z(\xi)\right|^2 = \left|\tilde{C}\right|^2 \exp\left(-\frac{1}{2}(\pi - i\sqrt{2}\rho)^2 - \frac{1}{2}(\pi + i\sqrt{2}\rho)^2\right) \\ &= |C|^2 \exp(2\rho^2)\exp\left(-\pi^2\right). \end{aligned} \tag{37}$$

  So here the situation is opposite: in position space the system is situated in $\xi = \pm\sqrt{2}\rho$ and momentum distribution is a Gaussian centered in $\pi = 0$.

Time dependence:

$$\begin{aligned} |z,t\rangle &= e^{-|z|^2/2}\sum_{n=0}\frac{z^n}{\sqrt{n!}}e^{-iE_nt/\hbar}|n\rangle \\ &= e^{-|z|^2/2}e^{-i\omega t/2}\sum_{n=0}\frac{z^n}{\sqrt{n!}}\left(e^{-i\omega t}\right)^n|n\rangle \\ &= e^{-i\omega t/2}|z(t)\rangle \end{aligned} \tag{38}$$

with

$$z(t) = ze^{-i\omega t}. \tag{39}$$

Assume

$$z = \rho e^{i\varphi} \tag{40}$$

then

$$z(t) = \rho e^{-i(\omega t - \varphi)} = \rho \cos(\omega t - \varphi) - i \sin(\omega t - \varphi) \tag{41}$$

and

$$\begin{aligned} \langle z, t| \, \hat{x} \, |z, t\rangle &= \sqrt{\frac{2\hbar}{m\omega}} \rho \cos(\omega t - \varphi) = x_0 \cos(\omega t - \varphi), \\ \langle z, t| \, \hat{p} \, |z, t\rangle &= -\sqrt{2\hbar m\omega} \rho \sin(\omega t - \varphi) = -p_0 \sin(\omega t - \varphi) \end{aligned} \tag{42}$$

with

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}} \rho, \; p_0 = \sqrt{2\hbar m\omega} \rho. \tag{43}$$

Note that this is motion of a classical oscillator. For semiclassical approximation we shall assume $\rho \gg 1$. Using (25) and (28) we have

$$\frac{\Delta x}{x_0} = \frac{1}{2\rho} \ll 1, \; \frac{\Delta p}{p_0} = \frac{1}{2\rho} \ll 1. \tag{44}$$

Relative uncertainties are time independent and very small for a semiclassical state.

Construction of a Schrödinger's cat

In time interval $[0, T]$ we switch on a "perturbation"

$$\hat{W} = \hbar g \left(\hat{a}^\dagger \hat{a} \right)^2. \tag{45}$$

Assume $g \gg \omega$ and $\omega T \ll 1$. This means

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 + \hat{W} \simeq \hat{W}. \tag{46}$$

Assume initial condition at time $t = 0$:

$$|\psi(0)\rangle = |z\rangle \,. \tag{47}$$

Since

$$\hat{W} \, |n\rangle = \hbar g n^2 \, |n\rangle \tag{48}$$

time dependence takes the following form

$$|\psi(t)\rangle = e^{-|z|^2/2} \sum_{n=0} \frac{z^n}{\sqrt{n!}} e^{-ign^2 t} \, |n\rangle \,. \tag{49}$$

Tis is rather complicated time dependence, but it simplifies for some particular values of time T:

- $T = 2\pi/g$

$$e^{-ign^2T} = 1$$

and

$$|\psi(T)\rangle = |z\rangle \, . \tag{50}$$

- $T = \pi/g$

$$e^{-ign^2T} = (-1)^n$$

since it is 1 for even n and -1 for odd n. Therefore

$$|\psi(T)\rangle = |-z\rangle \, . \tag{51}$$

- $T = \pi/2g$

$$\begin{aligned} e^{-ign^2T} &= e^{-in^2\pi/2} = \begin{cases} 1 & \text{for} \quad n \text{ - even} \\ -i & \text{for} \quad n \text{ - odd} \end{cases} \\ &= \frac{1}{2}\left[1 - i + (-)^n(1+i)\right] \\ &= \frac{1}{\sqrt{2}}\left(e^{-i\pi/4} + (-)^n e^{i\pi/4}\right) . \end{aligned} \tag{52}$$

In this case

$$\begin{aligned} |\psi(T)\rangle &= e^{-|z|^2/2}\frac{1}{\sqrt{2}}\sum_{n=0}\left(e^{-i\pi/4} + (-)^n e^{i\pi/4}\right)\frac{z^n}{\sqrt{n!}}\,|n\rangle \\ &= \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}\,|z\rangle + e^{i\pi/4}\,|-z\rangle\right) . \end{aligned} \tag{53}$$

Note that states $|z\rangle$ and $|-z\rangle$ are classically distinguishable for $z = \rho$ since average positions differ by a sign and for large ρ are therefore antinomic. They are therefore good models for Schrödinger's cat being *live* or *dead*. For $z = i\rho$ mean position is $\bar{x} = 0$, however two states $|z\rangle$ and $|-z\rangle$ have opposite velocities (momenta).

We shall calculate probability $P(\xi)$ and $P(\pi)$. In configuration space

$$\begin{aligned} P(\xi) &\sim \left|e^{-i\pi/4}\psi_z(\xi) + e^{i\pi/4}\psi_{-z}(\xi)\right|^2 \\ &= |\psi_z(\xi)|^2 + \left|\psi_{-z}(\xi)\right|^2 + e^{i\pi/2}\psi_z^*(\xi)\psi_{-z}(\xi) + e^{-i\pi/2}\psi_{-z}^*(\xi)\psi_z(\xi) \end{aligned} \tag{54}$$

where

$$\begin{aligned} |\psi_z(\xi)|^2 &= |C|^2 \exp\left(-\frac{1}{2}(\xi - \sqrt{2}z^*)^2 - \frac{1}{2}(\xi - \sqrt{2}z)^2\right) \\ &= |C|^2 \exp\left(-\frac{1}{2}(\xi^2 - 2\sqrt{2}\xi z^* + 2z^{*2}) - \frac{1}{2}(\xi^2 - 2\sqrt{2}\xi z + 2z^2)\right) \\ &= |C|^2 \exp\left(-\xi^2 + \sqrt{2}\xi(z^* + z) - (z^{*2} + z^2)\right) . \end{aligned} \tag{55}$$

In momentum space $z \to -iz$ and $\xi \to \pi$:

$$\left|\tilde{\psi}_z(\pi)\right|^2 = \left|\tilde{C}\right|^2 \exp\left(-\pi^2 + i\sqrt{2}\pi(z^* - z) + (z^{*2} + z^2)\right) \tag{56}$$

Interference term in configuration space can be obtained from (55) by replacing $z \to -z$:

$$\psi_z^*(\xi)\psi_{-z}(\xi) = |C|^2 \exp\left(-\xi^2 + \sqrt{2}\xi(z^* - z) - (z^{*2} + z^2)\right). \tag{57}$$

Interference term in momentum space reads

$$\tilde{\psi}_z^*(\pi)\tilde{\psi}_{-z}(\pi) = \left|\tilde{C}\right|^2 \exp\left(-\pi^2 + i\sqrt{2}\pi(z^* + z) + (z^{*2} + z^2)\right). \tag{58}$$

Momentum cat

Now we shall use $z = i\rho$. Let's first compute the probability distribution in momentum space:

$$\begin{aligned}
\left|\tilde{\psi}_{i\rho}(\pi)\right|^2 &= \left|\tilde{C}\right|^2 \exp\left(-\pi^2 + 2\sqrt{2}\pi\rho - 2\rho^2\right) \\
&= \left|\tilde{C}\right|^2 \exp\left(-\left(\pi - \sqrt{2}\rho\right)^2\right), \\
\left|\tilde{\psi}_{-i\rho}(\pi)\right|^2 &= \left|\tilde{C}\right|^2 \exp\left(-\left(\pi + \sqrt{2}\rho\right)^2\right).
\end{aligned} \tag{59}$$

These distributions have been already computed in Eqs. (35) and correspond to two largely separated Gaussians centred at $\pi = \pm\sqrt{2}\rho$. However, we still need to compute the interference term:

$$\tilde{\psi}_{i\rho}^*(\pi)\tilde{\psi}_{-i\rho}(\pi) = \left|\tilde{C}\right|^2 \exp\left(-\pi^2 - 2\rho^2\right). \tag{60}$$

We see that the interference term is almost zero because Gauss distribution is small for large $\pm\rho$. Therefore

$$P(\pi) \sim \exp\left(-\left(\pi - \sqrt{2}\rho\right)^2\right) + \exp\left(-\left(\pi + \sqrt{2}\rho\right)^2\right) \tag{61}$$

is a superposition of two separated Gaussians.

In configuration space the situation is different. We have two squares of $\psi_{\pm i\rho}$ already computed in (34)

$$\left|\psi_{\pm i\rho}(\xi)\right|^2 = |C|^2 \exp\left(-\xi^2 + 2\rho^2\right) \tag{62}$$

and

$$\psi_z^*(\xi)\psi_{-z}(\xi) = |C|^2 \exp\left(-\xi^2 + 2\rho^2 - i2\sqrt{2}\xi\rho\right) \tag{63}$$

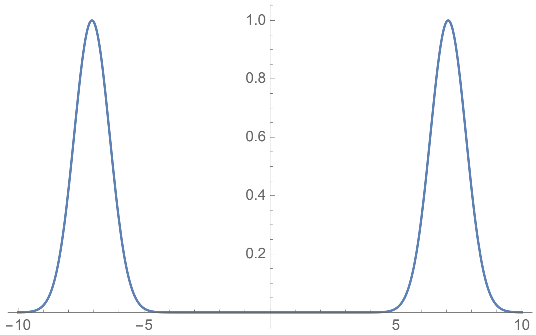

Figure 1: Probability in momentum space.

Hence

$$\begin{aligned} P(\xi) &\sim \exp\left(-\xi^2+2\rho^2\right)\left[2+\exp\left(-i2\left(\sqrt{2}\xi\rho-\frac{\pi}{4}\right)\right)+\exp\left(i2\left(\sqrt{2}\xi\rho-\frac{\pi}{4}\right)\right)\right] \\ &= 2\exp\left(-\xi^2+2\rho^2\right)\left[1+\cos\left(2\left(\sqrt{2}\xi\rho-\frac{\pi}{4}\right)\right)\right] \\ &= 4\exp\left(-\xi^2+2\rho^2\right)\cos^2\left(\sqrt{2}\xi\rho-\frac{\pi}{4}\right). \end{aligned} \tag{64}$$

We see that probability distribution in position space is not a Gaussian like in Eq. (34), but an oscillating function in a Gaussian envelope.

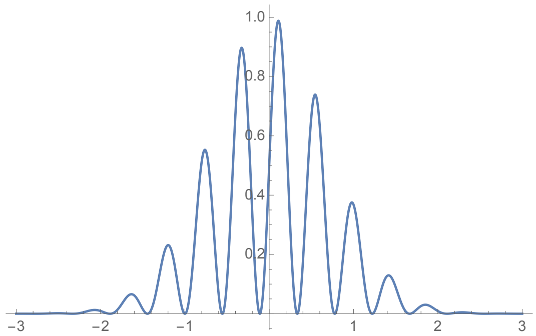

Figure 2: Probability in configuration space.

Position cat

Let's take $z=\rho$. Then

$$\begin{aligned} \left|\psi_\rho(\xi)\right|^2 &= |C|^2\exp\left(-\xi^2+2\sqrt{2}\xi\rho-2\rho^2\right)=|C|^2\exp\left(-(\xi-\sqrt{2}\rho)^2\right) \\ \left|\psi_{-\rho}(\xi)\right|^2 &= |C|^2\exp\left(-(\xi+\sqrt{2}\rho)^2\right). \end{aligned} \tag{65}$$

The interference term

$$\psi_z^*(\xi)\psi_{-z}(\xi)=|C|^2\exp\left(-\xi^2-2\rho^2\right) \tag{66}$$

and its complex conjugate are small in the large ρ limit. Therefore the probability distribution is the sum of two Gausses centred around $\pm\sqrt{2}\rho$, hence of a two antinomic states.

In momentum space

$$\left|\tilde{\psi}_{\pm\rho}(\pi)\right|^2 = \left|\tilde{C}\right|^2 \exp\left(-\pi^2 + 2\rho^2\right). \tag{67}$$

However we cannot neglect the interference term (changing notation for momentum to p):

$$\begin{aligned} e^{i\pi/2}\tilde{\psi}_z^*(p)\tilde{\psi}_{-z}(p) &= \left|\tilde{C}\right|^2 \exp\left(-p^2 + i2\sqrt{2}p\rho + 2\rho^2 + i\pi/2\right) \\ &= \left|\tilde{C}\right|^2 \exp\left(-p^2 + 2\rho^2\right) \exp(2i(\sqrt{2}p\rho + \pi/4)) \end{aligned} \tag{68}$$

And c.c.

$$e^{-i\pi/2}\tilde{\psi}_z(p)\tilde{\psi}_{-z}^*(p) = \left|\tilde{C}\right|^2 \exp\left(-p^2 + 2\rho^2\right) \exp(-2i(\sqrt{2}p\rho + \pi/4)). \tag{69}$$

Therefore full probability reads

$$\begin{aligned} P(p) &\sim \exp\left(-p^2 + 2\rho^2\right) \left[1 + \cos\left(2(\sqrt{2}p\rho + \pi/4)\right)\right] \\ &= \exp\left(-p^2 + 2\rho^2\right) \cos^2(\sqrt{2}p\rho + \pi/4). \end{aligned} \tag{70}$$

Hence momentum probability is an oscillationg function eneveloped by a Gaussian. In the case of statistical mixture it would be just a Gaussian.

Schrödinger's cat vs. statistical superposition

Can one distinguish superposition (53) from a statistical mixture of states $|z\rangle$ and $|-z\rangle$? In order to measure momenta we have to have resolution δp such that

$$\delta p \ll p_0\,, \tag{71}$$

where p_0 is the amplitude of oscillations in momentum space (43).

Consider simple pendulum of $m = 1$ g and 1 m length. Then

$$\omega = \sqrt{\frac{g}{l}} = 3.13\ \frac{1}{\text{s}}. \tag{72}$$

Let's assume that at time $t = 0$ pendulum is 1 μm from equlibrium:

$$x_0 = \sqrt{\frac{2\hbar}{m\omega}}\rho \quad \to \quad \rho = \sqrt{\frac{m\omega}{2\hbar}}x_0 = \sqrt{\frac{3.13}{2\times 1.054}10^{34}}\sqrt{\frac{\text{g/s}}{\text{J s}}}\mu\text{m} = 3.85\times 10^9. \tag{73}$$

Remember that J=kg m^2/s^2 = 10^{15}g μm^2/s^2 and $\hbar = 1.054 \times 10^{-34}$J s. Fom this we have that uncertainty is

$$\frac{\Delta x}{x_0} = \frac{1}{2\rho} \times 10^{-10}. \tag{74}$$

So indeed this is a classical state since quantum uncertainties are much smaller than the amplitudes of classical oscillations.

For the momentum distribution

$$\begin{aligned} p_0 &= \sqrt{2\hbar m\omega\rho} = \sqrt{2 \times 1.054 \times 10^{-34\times 10} \times 3.13}\sqrt{10^3 \text{g m}^2/\text{s} \times 1/\text{s}} \times 3.85 \times 10^9 \\ &= 3.13 \times 10^{-6} \frac{\text{g m}}{\text{s}}. \end{aligned}$$

This requires spacial resolution better than 1 μm, which is reasonable, given the initial condition. In order to resolve spacial oscillations one needs ξ resolution better than a distance corresponding to $\sqrt{2}\xi\rho \sim \pi$ (43):

$$\delta\xi \ll \frac{\pi}{\sqrt{2}\rho} \tag{75}$$

which translates for x

$$\delta x \ll \sqrt{\frac{\hbar}{m\omega}} \frac{\pi}{\sqrt{2}\rho} = \sqrt{\frac{1.054 \times 10^{-34}}{10^{-3} 3.13}} \sqrt{\frac{\text{kg m}^2/\text{s}}{\text{kg/s}}} \frac{\pi}{\sqrt{2} 3.85 \times 10^9} = 10^{-25} \text{ m}. \tag{76}$$

Such resolution is impossible to attain in practice.

Theoretically, however, a statistical ensemble of states $|z\rangle$ and $|-z\rangle$ would give the same momentum distribution as (53), however a competely different spacial distribution. In the first case the distribution is simply a Gaussian, and in the latter case a Gaussian enveloping the oscillations.

Fragility of a quantum superposition

Assume that the oscillator is in some way coupled with an (non-thermal) environment, whose quantum state will be denoted as $|\chi\rangle$. We shall try to estimate how long the system will stay in a superposition state (53). Let us first consider coupling of a coherent state. Initially at $t = 0$ the sytem is in a state $|\Phi(0)\rangle$

$$|\Phi(0)\rangle = |z(0)\rangle \, |\chi(0)\rangle \,, \tag{77}$$

Assume that time evolution is now modified:

$$z(t) \to z_\gamma(t) = z(t) e^{-\gamma t} \tag{78}$$

where $z(t)$ corresponds to (39). So in time t the state is now

$$|\Phi(t)\rangle = \left|z(t) e^{-\gamma t}\right\rangle |\chi(t)\rangle \,. \tag{79}$$

This means that the energy of an oscillator part of such a state is now

$$E_{\rm osc} = \hbar\omega \left(|z|^2 e^{-2\gamma t} + \frac{1}{2} \right). \tag{80}$$

After time much longer than $1/\gamma$ the system goes to a ground state. The energy gained by environment is therefore

$$\Delta E(t) = |z|^2 (1 - e^{-2\gamma t}) \simeq 2\gamma t \, |z|^2 , \tag{81}$$

where the last equality holds for short times $2\gamma t \ll 1$.

Let us now couple Schrödinger's cat state with the environment

$$|\Phi(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{-i\pi/4} \, |z_\gamma(t)\rangle \left|\chi^{(+)}(t)\right\rangle + e^{i\pi/4} \, |-z_\gamma(t)\rangle \left|\chi^{(-)}(t)\right\rangle \right), \tag{82}$$

where $\left|\chi^{(\pm)}(t)\right\rangle$ are two normalized states of the environment that are a'priori different (but not orthogonal). Let's choose again $z = i\rho$ with ρ being large. Then

$$P(x) = \frac{1}{2} \left[\left|\psi_{z_\gamma}(x)\right|^2 + \left|\psi_{-z_\gamma}(x)\right|^2 + 2\mathbf{Re} \left(i\psi^*_{z_\gamma}(x)\psi_{-z_\gamma}(x) \right) \left\langle \chi^{(+)}(t) \,\middle|\, \chi^{(-)}(t) \right\rangle \right], \tag{83}$$

where we have used that $e^{i\pi/2} = i$ and assumed that

$$\left\langle \chi^{(+)}(t) \,\middle|\, \chi^{(-)}(t) \right\rangle = \eta \in \mathcal{R}, \quad 0 < \eta < 1. \tag{84}$$

Going back to the dimensionless variables we see that the probability distribution in the configuration space for $z = i\rho$ reads

$$P(\xi) = 2 \exp\left(2(\rho e^{-\gamma t})^2\right) \exp\left(-\xi^2\right) \left[1 + \eta \cos\left(2 \left(\sqrt{2}\xi(\rho e^{-\gamma t}) - \frac{\pi}{4} \right) \right) \right] \tag{85}$$

and still has the Gaussian envelope, but the oscillatory term is suppressed by η. One can in principle still see the quantum wiggles in a position distribution if η is not too small. Momentum space distribution does not change, because the interference term did not contribute. One recovers two peaks centered at $\pm\rho e^{-\gamma t}\sqrt{2m\hbar w}$.

Assume now that the environment is represented by a harmonic oscillator of the same mass and frequency. Assume that initially the environment is in a ground state

$$|\chi\,(0)\rangle = |0\rangle\,.$$

If the coupling between the two oscillators is quadratic (as in $\hat{W}$) we will assume that in the course of time

- $\left|\chi^{(\pm)}(t)\right\rangle$ are coherent states $\left|\chi^{(\pm)}(t)\right\rangle = |\pm y(t)\rangle$
- and for short times $|y(t)|^2 = 2\gamma t \, |z(t)|^2$

Then

$$\eta = \left\langle \chi^{(+)}(t) \,\middle|\, \chi^{(-)}(t) \right\rangle = e^{-|y|^2} \sum_n \frac{1}{n!} y^{*n} (-y)^n = e^{-2|y|^2} \tag{86}$$

If we want η not too small $|y|^2 < 1$. For short times the energy of the first oscillator

$$E(t) = E(0) - 2\hbar\omega\gamma t \, |z(t)|^2 \tag{87}$$

and of the second

$$E'(t) = \hbar\omega \left(2\gamma t \, |z(t)|^2 + \frac{1}{2} \right), \tag{88}$$

where $2\gamma t \, |z(t)|^2$ is the energy gain of the second oscillator (remember that initially at $t = 0$ the oscillator was in the ground state). Total energy is conserved. Once the energy is transferred from the first oscillator to the second, the first oscillator becomes less and less semiclassical. Suppose that $1/(2\gamma) = 1$ year $= 3 \times 10^7$s, the time to reach $|y|^2 = 1$ is equal to:

$$t = \frac{1}{2\gamma} \frac{1}{\rho^2} = \frac{3 \times 10^7}{(3.85 \times 10^9)^2} \text{s} = 2 \times 10^{-12} \text{s}. \tag{89}$$

To conclude:

- Even for a system protected from the environment the quantum superpositions of macroscopic states are not observable,
- Interaction with the environment will very quickly destroy superposition;
- Attempts on small systems with a limited number of degrees of freedom have been undertaken, but are inconclusive.

based on J.-L. Basdevant, L. Dalibard *The Quantum Mechanics Solver*