# 1 Bell's inequalities [1]

One of the most difficult conceptual problems of QM is so called *collaps of the wave function.* Suppose that a spinless paricle at rest decays into two massive spin 1/2 particles that fly apart in opposite directions. After some time one measures spin projection of one of the decay products along the quantization axis $z$ (which may be chosen *e.g.* along the direction of motion). In principle two results of such a measurement are possible, namely $\pm 1/2$. Once this measurement is performed, the result of the measurement of second particle's spin, that maybe kilometers away, *has to be opposite*, despite the fact that no information can be transmitted from the first measurement point to th second one. This paradox has been pointed out by Einstein, Rosen and Podolsky, and led Einstein to mistrust probabilistic interpretation of QM. One may imagine that the probablistic nature of QM is a result of our limitted knowledge of all degrees of freedom of the system in question. Such degrees of freedom are often called *hidden variables.* Bell has proven that if such hidden variables existed one might propose an experiment that would allow to distinguish QM from QM with hidden variables.

## 1.1 Electron spin

Consider unit vector $\vec{n}_\theta$ in $x-z$ plane that has the following form:

$$\vec{n}_\theta = \cos\theta\,\vec{n}_z + \sin\theta\,\vec{n}_x \tag{1}$$

where $\vec{n}_{x,z}$ are unit vectors along $x$ an $z$ axis respectively. It is easy to show that the eigen values of the spin operator

$$S_\theta = \vec{n}_\theta \cdot \vec{S} \tag{2}$$

where $\vec{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ is spin 1/2 operator are $\pm 1/2$ (we keep $\hbar = 1$). Indeed

$$S_\theta = \frac{1}{2}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{3}$$

and the eigen-equation:

$$\begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0 \tag{4}$$

reads

$$-\cos^2\theta + \lambda^2 - \sin^2\theta = \lambda^2 - 1 = 0. \tag{5}$$

We can arrive at the same conclusion by calculating

$$\begin{aligned} S_\theta^2 &= \frac{1}{4}\left(\vec{n}_\theta \cdot \vec{\sigma}\right)\left(\vec{n}_\theta \cdot \vec{\sigma}\right) = \frac{1}{4}\left(n_{\theta\,i}\, n_{\theta\,j}\right)\left(\sigma_i \sigma_j\right) \\ &= \frac{1}{4}\left(n_{\theta\,i}\, n_{\theta\,j}\right)\left(\delta_{ij}\mathbf{1} + i\varepsilon_{ijk}\sigma_k\right) = \frac{1}{4}\mathbf{1}, \end{aligned} \tag{6}$$

[1] J.-L. Basdevant, L. Dalibard *The Quantum Mechanics Solver*

which means that the eigen-values are $\pm 1/2$.

The eigen-vectors can be calculated as follows:

$$\begin{bmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \tag{7}$$

which reduces to one equation

$$(\cos\theta - \lambda)\, x + \sin\theta\, y = 0. \tag{8}$$

To solve this let's use

$$\begin{aligned} \cos\theta &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}, \\ \sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}. \end{aligned} \tag{9}$$

Let's apply this to $\lambda = 1$:

$$\begin{aligned} &\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) x + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, y \\ = \;& -2\sin^2\frac{\theta}{2}\, x + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, y = 0. \end{aligned} \tag{10}$$

The solution reads:

$$x = \cos\frac{\theta}{2},\; y = \sin\frac{\theta}{2}. \tag{11}$$

For $\lambda = -1$:

$$\begin{aligned} &\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) x + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, y \\ = \;& 2\cos^2\frac{\theta}{2}\, x + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, y = 0 \end{aligned} \tag{12}$$

and the solution reads:

$$x = -\sin\frac{\theta}{2},\; y = \cos\frac{\theta}{2}. \tag{13}$$

Let's denote these eigen-vetors as follows:

$$\begin{aligned} |+\rangle_\theta &= +\cos\frac{\theta}{2}\,|+\rangle + \sin\frac{\theta}{2}\,|-\rangle\,, \\ |-\rangle_\theta &= -\sin\frac{\theta}{2}\,|+\rangle + \cos\frac{\theta}{2}\,|-\rangle\,. \end{aligned} \tag{14}$$

Let's assume that electron is initially in the state $|+\rangle_\theta$. Then we measure its spin along a different axis characterized by vector $\vec{n}_\alpha$. Again eigen-values of the operator $S_\alpha$

are $\pm 1/2$. Let's calculate corresponding probabilities:

$$\begin{aligned}
{}_\alpha\langle + \,|+\rangle_\theta &= \cos\frac{\alpha}{2}\cos\frac{\theta}{2} + \sin\frac{\alpha}{2}\sin\frac{\theta}{2} \\
&= \cos\frac{\theta-\alpha}{2}, \qquad (15) \\
{}_\alpha\langle - \,|+\rangle_\theta &= -\sin\frac{\alpha}{2}\cos\frac{\theta}{2} + \cos\frac{\alpha}{2}\sin\frac{\theta}{2} \\
&= \sin\frac{\theta-\alpha}{2}. \qquad (16) \\
{}_\alpha\langle - \,|-\rangle_\theta &= \sin\frac{\alpha}{2}\sin\frac{\theta}{2} + \cos\frac{\alpha}{2}\cos\frac{\theta}{2} \\
&= \cos\frac{\theta-\alpha}{2} \qquad (17)
\end{aligned}$$

Hence

$$\begin{aligned}
P_+(\alpha) &= {}_\alpha\langle + \,|+\rangle_\theta^2 = \cos^2\frac{\theta-\alpha}{2}, \\
P_-(\alpha) &= {}_\alpha\langle - \,|+\rangle_\theta^2 = \sin^2\frac{\theta-\alpha}{2}. \qquad (18)
\end{aligned}$$

From Eqs.(15,16) we have

$$\begin{aligned}
|+\rangle_\theta &= +\cos\frac{\theta-\alpha}{2}\,|+\rangle_\alpha + \sin\frac{\theta-\alpha}{2}\,|-\rangle_\alpha \\
|-\rangle_\theta &= -\sin\frac{\theta-\alpha}{2}\,|+\rangle_\alpha + \cos\frac{\theta-\alpha}{2}\,|-\rangle_\alpha \qquad (19)
\end{aligned}$$

It is now straigthforward to calculate the expectation value

$${}_\theta\langle +|\, S_\alpha \,|+\rangle_\theta = \frac{1}{2}\left(\cos^2\frac{\theta-\alpha}{2} - \sin^2\frac{\theta-\alpha}{2}\right) = \frac{1}{2}\cos(\theta-\alpha). \qquad (20)$$

It is easy to show that

$${}_\theta\langle -|\, S_\alpha \,|-\rangle_\theta = -\frac{1}{2}\cos(\theta-\alpha). \qquad (21)$$

We will also need

$$\begin{aligned}
{}_\theta\langle -|\, S_\alpha \,|+\rangle_\theta &= \frac{1}{2}\left(-\sin\frac{\theta-\alpha}{2}\cos\frac{\theta-\alpha}{2} - \sin\frac{\theta-\alpha}{2}\cos\frac{\theta-\alpha}{2}\right) \\
&= -\frac{1}{2}\sin(\theta-\alpha). \qquad (22)
\end{aligned}$$

## 1.2 Correlations between two spins

Consider a hydrogen atom that dissociates into a proton and electron. Assume that the electron-proton system is in the factorized spin state

$$|e:+\rangle_\theta\,|p:-\rangle_\theta \qquad (23)$$

Suppose we measure now electron spin along axis $\vec{n}_\alpha$, with the pertinent operator denoted as $S^e_\alpha$. Obviously the probability of finding $+1/2$ is the same as previously

$$P^e_+(\alpha) = \cos^2\frac{\theta-\alpha}{2},$$

however after the measurement the system is now in a state

$$|e:+\rangle_\alpha\,|p:-\rangle_\theta\,. \tag{24}$$

The proton spin remains unaffected by this measurement, beacsue the intial state was factorized.

It is easy to calculate expectation values of the spin operators $S^e_\alpha$ and $S^p_\beta$ in state (23):

$$\begin{aligned}\langle S^e_\alpha\rangle &= {}_\theta\langle e:+|\,S^e_\alpha\,|e:+\rangle_\theta\;{}_\theta\langle p:-|\,\mathbf{1}^p\,|p:-\rangle_\theta \\ &= \frac{1}{2}\cos(\theta-\alpha)\end{aligned} \tag{25}$$

and

$$\begin{aligned}\left\langle S^p_\beta\right\rangle &= {}_\theta\langle e:+|\,\mathbf{1}^e\,|e:+\rangle_\theta\;{}_\theta\langle p:-|\,S^p_\beta\,|p:-\rangle_\theta \\ &= -\frac{1}{2}\cos(\theta-\beta).\end{aligned} \tag{26}$$

Finally we can also quite easily calculate

$$\begin{aligned}\left\langle S^e_\alpha\otimes S^p_\beta\right\rangle &= {}_\theta\langle e:+|\,S^e_\alpha\,|e:+\rangle_\theta\;{}_\theta\langle p:-|\,S^p_\beta\,|p:-\rangle_\theta \\ &= -\frac{1}{4}\cos(\theta-\alpha)\cos(\theta-\beta).\end{aligned} \tag{27}$$

With these results and (6) we can compute the correlation coefficient

$$E(\alpha,\beta) = \frac{\left\langle S^e_\alpha\otimes S^p_\beta\right\rangle - \langle S^e_\alpha\rangle\left\langle S^p_\beta\right\rangle}{\sqrt{\langle S^{e\,2}_\alpha\rangle\left\langle S^{p\,2}_\beta\right\rangle}} = 0.$$

This result reflects the fact that the system was in a factorized state.

## 1.3 Correlations in the singlet state

Let's assume that after the dissociation proton and electron are in the singlet state:

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|e:+\rangle\,|p:-\rangle - |e:-\rangle\,|p:+\rangle\right). \tag{28}$$

Suppose we measure $S^e_\alpha$, what are the possible results and their probabilities. To this end let us decompose electron unity operator in the basis of the eigenstates of $S^e_\alpha$:

$$\mathbf{1} = |e:+\rangle_{\alpha\,\alpha}\langle e:+|\otimes\mathbf{1}^p \;+\; |e:-\rangle_{\alpha\,\alpha}\langle e:-|\otimes\mathbf{1}^p \tag{29}$$

and apply it to the state (28):

$$|0\rangle = \frac{1}{\sqrt{2}} [|e:+\rangle_{\alpha\,\alpha} \langle e:+|\, e:+\rangle \, |p:-\rangle - |e:+\rangle_{\alpha\,\alpha} \langle e:+|\, e:-\rangle \, |p:+\rangle + |e:-\rangle_{\alpha\,\alpha} \langle e:-|\, e:+\rangle \, |p:-\rangle - |e:-\rangle_{\alpha\,\alpha} \langle e:-|\, e:-\rangle \, |p:+\rangle] \quad (30)$$

and use Eqs.(15-17)

$$|0\rangle = \frac{1}{\sqrt{2}} \left[ \cos\frac{\alpha}{2} |e:+\rangle_\alpha \, |p:-\rangle - \sin\frac{\alpha}{2} |e:+\rangle_\alpha \, |p:+\rangle - \sin\frac{\alpha}{2} |e:-\rangle_{\alpha\,\alpha} |p:-\rangle - \cos\frac{\alpha}{2} |e:-\rangle_\alpha \, |p:+\rangle \right]. \quad (31)$$

Possible results are

- $+1/2$ with probability

$$P_+(\alpha) = \frac{1}{2}\cos^2\frac{\alpha}{2} + \frac{1}{2}\sin^2\frac{\alpha}{2} = \frac{1}{2} \quad (32)$$

- $-1/2$ with probability

$$P_-(\alpha) = \frac{1}{2}$$

  as well.

Let's assume that the result of the measurement was $+1/2$. This means that after the measurement the wave function *collapsed* to

$$|0\rangle \underset{S^e_\alpha = +\frac{1}{2}}{\rightarrow} |\Phi\rangle = \cos\frac{\alpha}{2} |e:+\rangle_\alpha \, |p:-\rangle - \sin\frac{\alpha}{2} |e:+\rangle_\alpha \, |p:+\rangle = |e:+\rangle_\alpha \, |p:-\rangle_\alpha \, . \quad (33)$$

If we now measure $S^p_\beta$ we have again two possible results $\pm 1/2$ with the following probabilities

$$P_+(\beta) = |\,_\beta\langle + \,|-\rangle_\alpha| = \sin^2\frac{\alpha-\beta}{2},$$
$$P_-(\beta) = |\,_\beta\langle - \,|-\rangle_\alpha| = \cos^2\frac{\alpha-\beta}{2}. \quad (34)$$

Let us summarize: in the first measurement of electron spin we get both possible results with the same probability 1/2 but the probabilities to get $\pm 1/2$ in the second measurement of the proton spin depend on the relative angle between the measurerement axes and are in general not equal. Had we measuered the proton spin first, we would have obtained $\pm 1/2$ with equal probabilities 1/2. This difference was unnaceptable for Einstein who claimed that "the real states of two spatially separated objects must be independent of one another". This simple example leads to a conclusion that QM is not a local theory as far as mesurement is concerned. This non-locality, however, does not allow for the instantenous transmition of information. In each single measurement of the proton spin we cannot tell whether the electron spin has been measured before. One needs a series of experiments on the same state to find this non-local character of QM.

Let us now calculate correlation coefficient $E(\alpha, \beta)$. Let's start from the averages:

$$\begin{aligned} \langle 0| S^e_\alpha |0\rangle &= \frac{1}{2} \langle e:+| S^e_\alpha |e:+\rangle + \frac{1}{2} \langle e:-| S^e_\alpha |e:-\rangle \\ &= \frac{1}{2} (\cos\alpha - \cos\alpha) = 0 \end{aligned} \tag{35}$$

where we have used (20) and (21). The same result holds for the proton

$$\langle 0| S^p_\beta |0\rangle = 0. \tag{36}$$

Let us now calculate

$$\begin{aligned} \langle 0| S^e_\alpha \otimes S^p_\beta |0\rangle &= \frac{1}{2} \big(\langle +| S^e_\alpha |+\rangle \langle -| S^p_\beta |-\rangle - \langle +| S^e_\alpha |-\rangle \langle -| S^p_\beta |+\rangle \\ &\quad - \langle -| S^e_\alpha |+\rangle \langle +| S^p_\beta |-\rangle + \langle -| S^e_\alpha |-\rangle \langle +| S^p_\beta |+\rangle\big) \\ &= \frac{1}{8} 2\,(-\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ &= -\frac{1}{4}\cos(\alpha-\beta) \end{aligned} \tag{37}$$

where we have used (20-22). Hence

$$E(\alpha,\beta) = \frac{-\frac{1}{4}\cos(\alpha-\beta)}{\sqrt{\frac{1}{4}\frac{1}{4}}} = -\cos(\alpha-\beta). \tag{38}$$