1 Bell's inequalities [1]

One of the most difficult conceptual problems of QM is so called *collaps of the wave function.* Suppose that a spinless paricle at rest decays into two massive spin 1/2 particles that fly apart in opposite directions. After some time one measures spin projection of one of the decay products along the quantization axis z (which may be chosen *e.g.* along the direction of motion). In principle two results of such a measurement are possible, namely $\pm 1/2$. Once this measurement is performed, the result of the measurement of second particle's spin, that maybe kilometers away, *has to be opposite*, despite the fact that no information can be transmitted from the first measurement point to th second one. This paradox has been pointed out by Einstein, Rosen and Podolsky, and led Einstein to mistrust probabilistic interpretation of QM. One may imagine that the probablistic nature of QM is a result of our limitted knowledge of all degrees of freedom of the system in question. Such degrees of freedom are often called *hidden variables.* Bell has proven that if such hidden variables existed one might propose an experiment that would allow to distinguish QM from QM with hidden variables.

1.1 Electron spin

Consider unit vector $\vec{n}_\theta$ in $x-z$ plane that has the following form:

$$\vec{n}_\theta = \cos\theta\,\vec{n}_z + \sin\theta\,\vec{n}_x \tag{1}$$

where $\vec{n}_{x,z}$ are unit vectors along x an z axis respectively. It is easy to show that the eigen values of the spin operator

$$S_\theta = \vec{n}_\theta \cdot \vec{S} \tag{2}$$

where $\vec{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ is spin 1/2 operator are $\pm 1/2$ (we keep $\hbar = 1$). Indeed

$$S_\theta = \frac{1}{2}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{3}$$

and the eigen-equation:

$$\begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{vmatrix} = 0 \tag{4}$$

reads

$$-\cos^2\theta + \lambda^2 - \sin^2\theta = \lambda^2 - 1 = 0. \tag{5}$$

We can arrive at the same conclusion by calculating

$$\begin{aligned} S_\theta^2 &= \frac{1}{4}\left(\vec{n}_\theta \cdot \vec{\sigma}\right)\left(\vec{n}_\theta \cdot \vec{\sigma}\right) = \frac{1}{4}\left(n_{\theta\, i}\, n_{\theta\, j}\right)\left(\sigma_i \sigma_j\right) \\ &= \frac{1}{4}\left(n_{\theta\, i}\, n_{\theta\, j}\right)\left(\delta_{ij}\mathbf{1} + i\varepsilon_{ijk}\sigma_k\right) = \frac{1}{4}\mathbf{1}, \end{aligned} \tag{6}$$

[1] J.-L. Basdevant, L. Dalibard *The Quantum Mechanics Solver*

which means that the eigen-values are $\pm 1/2$.

The eigen-vectors can be calculated as follows:

$$\left[\begin{array}{cc} \cos\theta - \lambda & \sin\theta \\ \sin\theta & -\cos\theta - \lambda \end{array} \right] \left[\begin{array}{c} x \\ y \end{array} \right] = 0 \tag{7}$$

which reduces to one equation

$$(\cos\theta - \lambda)\, x + \sin\theta\, y = 0. \tag{8}$$

To solve this let's use

$$\begin{aligned} \cos\theta &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}, \\ \sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}. \end{aligned} \tag{9}$$

Let's apply this to $\lambda = 1$:

$$\begin{aligned} & \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) x + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, y \\ = & -2\sin^2\frac{\theta}{2}\, x + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, y = 0. \end{aligned} \tag{10}$$

The solution reads:

$$x = \cos\frac{\theta}{2},\ y = \sin\frac{\theta}{2}. \tag{11}$$

For $\lambda = -1$:

$$\begin{aligned} & \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right) x + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, y \\ = & \ 2\cos^2\frac{\theta}{2}\, x + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, y = 0 \end{aligned} \tag{12}$$

and the solution reads:

$$x = -\sin\frac{\theta}{2},\ y = \cos\frac{\theta}{2}. \tag{13}$$

Let's denote these eigen-vetors as follows:

$$\begin{aligned} |+\rangle_\theta &= +\cos\frac{\theta}{2}\,|+\rangle + \sin\frac{\theta}{2}\,|-\rangle\,, \\ |-\rangle_\theta &= -\sin\frac{\theta}{2}\,|+\rangle + \cos\frac{\theta}{2}\,|-\rangle\,. \end{aligned} \tag{14}$$

Let's assume that electron is initially in the state $|+\rangle_\theta$. Then we measure its spin along a different axis characterized by vector $\vec{n}_\alpha$. Again eigen-values of the operator S_α

are $\pm 1/2$. Let's calculate corresponding probabilities:

$$\begin{aligned} {}_\alpha\langle + \,|+\rangle_\theta &= \cos\frac{\alpha}{2}\cos\frac{\theta}{2} + \sin\frac{\alpha}{2}\sin\frac{\theta}{2} \\ &= \cos\frac{\theta-\alpha}{2}, \end{aligned} \tag{15}$$

$$\begin{aligned} {}_\alpha\langle - \,|+\rangle_\theta &= -\sin\frac{\alpha}{2}\cos\frac{\theta}{2} + \cos\frac{\alpha}{2}\sin\frac{\theta}{2} \\ &= \sin\frac{\theta-\alpha}{2}. \end{aligned} \tag{16}$$

$$\begin{aligned} {}_\alpha\langle - \,|-\rangle_\theta &= \sin\frac{\alpha}{2}\sin\frac{\theta}{2} + \cos\frac{\alpha}{2}\cos\frac{\theta}{2} \\ &= \cos\frac{\theta-\alpha}{2} \end{aligned} \tag{17}$$

Hence

$$\begin{aligned} P_+(\alpha) &= {}_\alpha\langle + \,|+\rangle_\theta^2 = \cos^2\frac{\theta-\alpha}{2}, \\ P_-(\alpha) &= {}_\alpha\langle - \,|+\rangle_\theta^2 = \sin^2\frac{\theta-\alpha}{2}. \end{aligned} \tag{18}$$

From Eqs.(15,16) we have

$$\begin{aligned} |+\rangle_\theta &= +\cos\frac{\theta-\alpha}{2}\,|+\rangle_\alpha + \sin\frac{\theta-\alpha}{2}\,|-\rangle_\alpha \\ |-\rangle_\theta &= -\sin\frac{\theta-\alpha}{2}\,|+\rangle_\alpha + \cos\frac{\theta-\alpha}{2}\,|-\rangle_\alpha \end{aligned} \tag{19}$$

It is now straigthforward to calculate the expectation value

$${}_\theta\langle +|\, S_\alpha \,|+\rangle_\theta = \frac{1}{2}\left(\cos^2\frac{\theta-\alpha}{2} - \sin^2\frac{\theta-\alpha}{2}\right) = \frac{1}{2}\cos(\theta-\alpha). \tag{20}$$

It is easy to show that

$${}_\theta\langle -|\, S_\alpha \,|-\rangle_\theta = -\frac{1}{2}\cos(\theta-\alpha). \tag{21}$$

We will also need

$$\begin{aligned} {}_\theta\langle -|\, S_\alpha \,|+\rangle_\theta &= \frac{1}{2}\left(-\sin\frac{\theta-\alpha}{2}\cos\frac{\theta-\alpha}{2} - \sin\frac{\theta-\alpha}{2}\cos\frac{\theta-\alpha}{2}\right) \\ &= -\frac{1}{2}\sin(\theta-\alpha). \end{aligned} \tag{22}$$

1.2 Correlations between two spins

Consider a hydrogen atom that dissociates into a proton and electron. Assume that the electron-proton system is in the factorized spin state

$$|e:+\rangle_\theta\,|p:-\rangle_\theta \tag{23}$$

Suppose we measure now electron spin along axis $\vec{n}_\alpha$, with the pertinent operator denoted as S^e_α. Obviously the probability of finding $+1/2$ is the same as previously

$$P^e_+(\alpha) = \cos^2\frac{\theta-\alpha}{2},$$

however after the measurement the system is now in a state

$$|e:+\rangle_\alpha\,|p:-\rangle_\theta\,. \tag{24}$$

The proton spin remains unaffected by this measurement, beacsue the intial state was factorized.

It is easy to calculate expectation values of the spin operators S^e_α and S^p_β in state (23):

$$\begin{aligned}\langle S^e_\alpha\rangle &= {}_\theta\langle e:+|\,S^e_\alpha\,|e:+\rangle_\theta\;{}_\theta\langle p:-|\,\mathbf{1}^p\,|p:-\rangle_\theta \\ &= \frac{1}{2}\cos(\theta-\alpha)\end{aligned} \tag{25}$$

and

$$\begin{aligned}\left\langle S^p_\beta\right\rangle &= {}_\theta\langle e:+|\,\mathbf{1}^e\,|e:+\rangle_\theta\;{}_\theta\langle p:-|\,S^p_\beta\,|p:-\rangle_\theta \\ &= -\frac{1}{2}\cos(\theta-\beta).\end{aligned} \tag{26}$$

Finally we can also quite easily calculate

$$\begin{aligned}\left\langle S^e_\alpha\otimes S^p_\beta\right\rangle &= {}_\theta\langle e:+|\,S^e_\alpha\,|e:+\rangle_\theta\;{}_\theta\langle p:-|\,S^p_\beta\,|p:-\rangle_\theta \\ &= -\frac{1}{4}\cos(\theta-\alpha)\cos(\theta-\beta).\end{aligned} \tag{27}$$

With these results and (6) we can compute the correlation coefficient

$$E(\alpha,\beta) = \frac{\left\langle S^e_\alpha\otimes S^p_\beta\right\rangle - \langle S^e_\alpha\rangle\left\langle S^p_\beta\right\rangle}{\sqrt{\left\langle S^{e\,2}_\alpha\right\rangle\left\langle S^{p\,2}_\beta\right\rangle}} = 0.$$

This result reflects the fact that the system was in a factorized state.

1.3 Correlations in the singlet state

Let's assume that after the dissociation proton and electron are in the singlet state:

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|e:+\rangle\,|p:-\rangle - |e:-\rangle\,|p:+\rangle\right). \tag{28}$$

Suppose we measure S^e_α, what are the possible results and their probabilities. To this end let us decompose electron unity operator in the basis of the eigenstates of S^e_α:

$$\mathbf{1} = |e:+\rangle_{\alpha\,\alpha}\langle e:+|\otimes\mathbf{1}^p \;+\; |e:-\rangle_{\alpha\,\alpha}\langle e:-|\otimes\mathbf{1}^p \tag{29}$$

and apply it to the state (28):

$$
\begin{aligned}
|0\rangle &= \frac{1}{\sqrt{2}}\left[|e:+\rangle_{\alpha\,\alpha}\langle e:+|\, e:+\rangle\,|p:-\rangle - |e:+\rangle_{\alpha\,\alpha}\langle e:+|\, e:-\rangle\,|p:+\rangle\right. \\
&\left. + |e:-\rangle_{\alpha\,\alpha}\langle e:-|\, e:+\rangle\,|p:-\rangle - |e:-\rangle_{\alpha\,\alpha}\langle e:-|\, e:-\rangle\,|p:+\rangle\right] \qquad (30)
\end{aligned}
$$

and use Eqs.(15-17)

$$
\begin{aligned}
|0\rangle &= \frac{1}{\sqrt{2}}\left[\cos\frac{\alpha}{2}\,|e:+\rangle_{\alpha}\,|p:-\rangle - \sin\frac{\alpha}{2}\,|e:+\rangle_{\alpha}\,|p:+\rangle\right. \\
&\left. - \sin\frac{\alpha}{2}\,|e:-\rangle_{\alpha\,\alpha}\,|p:-\rangle - \cos\frac{\alpha}{2}\,|e:-\rangle_{\alpha}\,|p:+\rangle\right]. \qquad (31)
\end{aligned}
$$

Possible results are

- $+1/2$ with probability

$$
P_{+}(\alpha) = \frac{1}{2}\cos^2\frac{\alpha}{2} + \frac{1}{2}\sin^2\frac{\alpha}{2} = \frac{1}{2} \qquad (32)
$$

- $-1/2$ with probability

$$
P_{-}(\alpha) = \frac{1}{2}
$$

  as well.

Let's assume that the result of the measurement was $+1/2$. This means that after the measurement the wave function *collapsed* to

$$
\begin{aligned}
|0\rangle \underset{S^e_{\alpha}=+\frac{1}{2}}{\longrightarrow} |\Phi\rangle &= \cos\frac{\alpha}{2}\,|e:+\rangle_{\alpha}\,|p:-\rangle - \sin\frac{\alpha}{2}\,|e:+\rangle_{\alpha}\,|p:+\rangle \\
&= |e:+\rangle_{\alpha}\,|p:-\rangle_{\alpha}\,. \qquad (33)
\end{aligned}
$$

If we now measure S^p_{β} we have again two possible results $\pm 1/2$ with the following probabilities

$$
\begin{aligned}
P_{+}(\beta) &= |\,_{\beta}\langle +\,|-\rangle_{\alpha}| = \sin^2\frac{\alpha-\beta}{2}, \\
P_{-}(\beta) &= |\,_{\beta}\langle -\,|-\rangle_{\alpha}| = \cos^2\frac{\alpha-\beta}{2}. \qquad (34)
\end{aligned}
$$

Let us summarize: in the first measurement of electron spin we get both possible results with the same probability 1/2 but the probabilities to get $\pm 1/2$ in the second measurement of the proton spin depend on the relative angle between the measurerement axes and are in general not equal. Had we measuered the proton spin first, we would have obtained $\pm 1/2$ with equal probabilities 1/2. This difference was unnaceptable for Einstein who claimed that "the real states of two spatially separated objects must be independent of one another". This simple example leads to a conclusion that QM is not a local theory as far as mesurement is concerned. This non-locality, however, does not allow for the instantenous transmition of information. In each single measurement of the proton spin we cannot tell whether the electron spin has been measured before. One needs a series of experiments on the same state to find this non-local character of QM.

Let us now calculate correlation coefficient $E(\alpha, \beta)$. Let's start from the averages:

$$\begin{aligned} \langle 0| S^e_\alpha |0\rangle &= \frac{1}{2} \langle e:+| S^e_\alpha |e:+\rangle + \frac{1}{2} \langle e:-| S^e_\alpha |e:-\rangle \\ &= \frac{1}{2} (\cos\alpha - \cos\alpha) = 0 \end{aligned} \tag{35}$$

where we have used (20) and (21). The same result holds for the proton

$$\langle 0| S^p_\beta |0\rangle = 0. \tag{36}$$

Let us now calculate

$$\begin{aligned} \langle 0| S^e_\alpha \otimes S^p_\beta |0\rangle &= \frac{1}{2} \big(\langle +| S^e_\alpha |+\rangle \langle -| S^p_\beta |-\rangle - \langle +| S^e_\alpha |-\rangle \langle -| S^p_\beta |+\rangle \\ &\quad - \langle -| S^e_\alpha |+\rangle \langle +| S^p_\beta |-\rangle + \langle -| S^e_\alpha |-\rangle \langle +| S^p_\beta |+\rangle \big) \\ &= \frac{1}{8} 2 (-\cos\alpha\cos\beta - \sin\alpha\sin\beta) \\ &= -\frac{1}{4}\cos(\alpha-\beta) \end{aligned} \tag{37}$$

where we have used (20-22). Hence

$$E(\alpha,\beta) = \frac{-\frac{1}{4}\cos(\alpha-\beta)}{\sqrt{\frac{1}{4}\frac{1}{4}}} = -\cos(\alpha-\beta). \tag{38}$$

1.4 Simple hidden variable model

Here we propose a simple model of hidden variables. Assume that after the decay the system is in a factorized state (23)

$$|e:+\rangle_\theta \, |p:-\rangle_\theta \tag{39}$$

however direction $\vec{n}_\theta$ varies from event to event. In this case angle θ is a hidden variable. Assuming that all directions θ are equaly probable we define now the expectation value of an observable $\mathcal{O}$ as

$$\langle \mathcal{O} \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} \; {}_\theta\langle p:-|_\theta \langle e:+| \, \mathcal{O} \, |e:+\rangle_\theta \, |p:-\rangle_\theta \,. \tag{40}$$

Let's first calculate expectation value of the electron spin

$$\langle S^e_\alpha \rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{2} \cos(\theta-\alpha) = 0 \tag{41}$$

as in the case of the singlet state. The same applies to the proton spin. Therefore our simple hidden variable model reproduce in this case the results of QM. For double correlation we get however different result:

$$\left\langle S^e_\alpha \otimes S^p_\beta \right\rangle = -\int_0^{2\pi} \frac{d\theta}{2\pi}\frac{1}{2}\cos(\theta-\alpha)\frac{1}{2}\cos(\theta-\beta)$$

$$= -\frac{1}{4}\int_0^{2\pi} \frac{d\theta}{2\pi}\left(\cos\theta\cos\alpha + \sin\theta\sin\alpha\right)\left(\cos\theta\cos\beta + \sin\theta\sin\beta\right)$$

$$= -\frac{1}{4}\int_0^{2\pi} \frac{d\theta}{2\pi}\left[\cos^2\theta\cos\alpha\cos\beta + \cos\theta\sin\theta\left(\cos\alpha\sin\beta+\sin\alpha\cos\beta\right)+\sin^2\theta\sin\alpha\sin\beta\right].$$

Taking into account that

$$\int_0^{2\pi} \frac{d\theta}{2\pi}\cos^2\theta = \int_0^{2\pi} \frac{d\theta}{2\pi}\sin^2\theta = \frac{1}{2},$$

$$\int_0^{2\pi} \frac{d\theta}{2\pi}\cos\theta\sin\theta = 0 \tag{42}$$

we get

$$\left\langle S^e_\alpha \otimes S^p_\beta \right\rangle = -\frac{1}{8}\left(\cos\alpha\cos\beta+\sin\alpha\sin\beta\right) = -\frac{1}{8}\cos(\alpha-\beta). \tag{43}$$

Finally

$$E(\alpha,\beta) = \frac{-\frac{1}{8}\cos(\alpha-\beta)}{\sqrt{\frac{1}{4}\frac{1}{4}}} = -\frac{1}{2}\cos(\alpha-\beta) \tag{44}$$

is twice smaller than the QM result (38). Therefore by measuring E one can distinguish between QM and hidden variable model. Question arises whether it is just an accidental feature of our simple hidden variable model, or whether it is generally impossible to construct a hidden variable model that would reproduce E of QM.

1.5 Bell's theorem

Bell proved in 1965 that the disagreement between QM and hidden variable theories is very general. Consider hidden variable (λ) theory characterized by two functions

$$A(\lambda,\vec{n}_\alpha) = \pm\frac{1}{2},\ B(\lambda,\vec{n}_\beta) = \pm\frac{1}{2}\ \text{depending on}\ \lambda \tag{45}$$

corresponding to electron and proton spin, respectively. Note that B does not depend on $\vec{n}_\alpha$ and A does not depend on $\vec{n}_\beta$. This *locality* requirement is essential for further discussion. We shall assume that λ is distributed according to the probability distribution $P(\lambda) \geq 0$ such that

$$\int d\lambda\, P(\lambda) = 1 \tag{46}$$

and

$$\begin{aligned} \langle S^e_\alpha \rangle &= \int d\lambda\, P(\lambda) A(\lambda, \vec{n}_\alpha) = 0, \\ \langle S^p_\beta \rangle &= \int d\lambda\, P(\lambda) B(\lambda, \vec{n}_\beta) = 0 \end{aligned} \tag{47}$$

to be compatible with (35,36) and (41). Therefore in the framework of hidden variable theory

$$E(\alpha, \beta) = 4 \int d\lambda\, P(\lambda) A(\lambda, \vec{n}_\alpha) B(\lambda, \vec{n}_\beta). \tag{48}$$

We shall now define quantity S

$$S = E(\alpha, \beta) + E(\alpha, \beta') + E(\alpha', \beta') - E(\alpha', \beta) \tag{49}$$

and prove that in hidden variable theory for any angles $\alpha, \alpha', \beta, \beta'$

$$|S| \leq 2. \tag{50}$$

Equation (50) is called Bell's inequality. We will show that in QM for some choices of angles $\alpha, \alpha', \beta, \beta'$ it is possible that $|S| > 2$. To this end let us first show that

$$s = A(\lambda, \alpha) B(\lambda, \beta) + A(\lambda, \alpha) B(\lambda, \beta') + A(\lambda, \alpha') B(\lambda, \beta') - A(\lambda, \alpha') B(\lambda, \beta) = \pm\frac{1}{2}. \tag{51}$$

Indeed

$$s = A(\lambda, \alpha) \underbrace{[B(\lambda, \beta) + B(\lambda, \beta')]}_{\pm 1 \text{ or } 0} + A(\lambda, \alpha') \underbrace{[B(\lambda, \beta') - B(\lambda, \beta)]}_{0 \text{ or } \pm 1} \tag{52}$$

Note that A is either $1/2$ or $-1/2$. So whenever the first $[\ldots] \neq 0$

$$-\frac{1}{2} \leq \int d\lambda\, P(\lambda) A(\lambda, \alpha) \left[B(\lambda, \beta) + B(\lambda, \beta')\right] \leq \frac{1}{2} \tag{53}$$

and whenever the second $[\ldots] \neq 0$

$$-\frac{1}{2} \leq \int d\lambda\, P(\lambda) A(\lambda, \alpha') \left[B(\lambda, \beta') - B(\lambda, \beta)\right] \leq \frac{1}{2}. \tag{54}$$

Because E has in denominator $1/4$,we arrive at (50).

In QM

$$S = -\cos(\underbrace{\alpha - \beta}_{=\theta_1}) - \cos(\underbrace{\alpha - \beta'}_{=-\theta_2}) - \cos(\underbrace{\alpha' - \beta'}_{=\theta_3}) + \cos(\underbrace{\alpha' - \beta}_{=\theta_4}). \tag{55}$$

Note that

$$\theta_1 + \theta_2 + \theta_3 = \alpha - \beta + \beta' - \alpha + \alpha' - \beta' = \alpha' - \beta = \theta_4. \tag{56}$$

Therefore

$$S = -\cos\theta_1 - \cos\theta_2 - \cos\theta_3 + \cos(\theta_1 + \theta_2 + \theta_3). \tag{57}$$

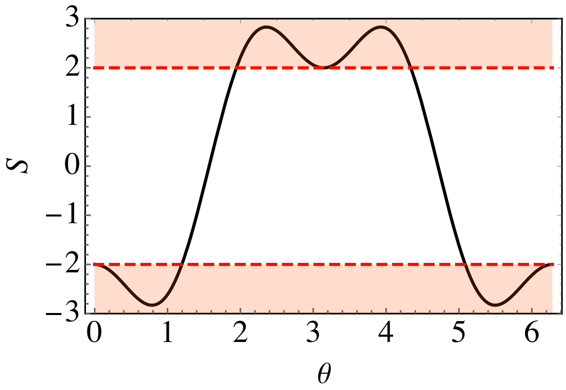
Let's make a speciphic coice of angles $\theta_{1,2,3}$:

$$\theta_1 = \theta_2 = \theta_3 \stackrel{\text{df}}{=} \theta \tag{58}$$

With this choice

$$S = -3\cos\theta + \cos(3\theta). \tag{59}$$

Let's plot $S(\theta)$:



We see that there are regions (shaded areas) where where Bell's inequality (50) is violated but allowed by QM.