Prove that:

$$\partial_\mu K^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \equiv \frac{1}{2} F F^*, \tag{1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu, \tag{2}$$

and

$$K^\mu = \varepsilon^{\mu\nu\rho\sigma}\left(A^a_\nu F^a_{\rho\sigma} - \frac{g}{3} f^{abc} A^a_\nu A^b_\rho A^c_\sigma\right). \tag{3}$$

First we rewrite the right hand side of (1) using (2):

$$\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f^{abc} A^b_\mu A^c_\nu\right) F^a_{\rho\sigma} \tag{4}$$

$$= \varepsilon^{\mu\nu\rho\sigma} \partial_\mu A^a_\nu F^a_{\rho\sigma} \tag{5}$$

$$+ \frac{g}{2}\varepsilon^{\mu\nu\rho\sigma} f^{abc} A^b_\mu A^c_\nu F^a_{\rho\sigma}. \tag{6}$$

Next we rewrite the left hand side of (1) using (3):

$$\partial_\mu K^\mu = \varepsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A^a_\nu F^a_{\rho\sigma} - \frac{g}{3} f^{abc} A^a_\nu A^b_\rho A^c_\sigma\right) \tag{7}$$

$$= \varepsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A^a_\nu\right) F^a_{\rho\sigma} \tag{8}$$

$$+ \varepsilon^{\mu\nu\rho\sigma} A^a_\nu \left(\partial_\mu F^a_{\rho\sigma}\right) - \frac{g}{3}\varepsilon^{\mu\nu\rho\sigma} f^{abc} \partial_\mu\left(A^a_\nu A^b_\rho A^c_\sigma\right). \tag{9}$$

We observe now that becuse of Eqs.(4)-(6) line (8) equals

$$\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\mu A^a_\nu\right) F^a_{\rho\sigma} = \frac{1}{2} F F^* - \frac{g}{2}\varepsilon^{\mu\nu\rho\sigma} f^{abc} A^b_\mu A^c_\nu F^a_{\rho\sigma}.$$

Therefore we have to show that

$$\underbrace{-\frac{g}{2}\varepsilon^{\mu\nu\rho\sigma} f^{abc} A^b_\mu A^c_\nu F^a_{\rho\sigma}}_{=C_1} + \underbrace{\varepsilon^{\mu\nu\rho\sigma} A^a_\nu\left(\partial_\mu F^a_{\rho\sigma}\right)}_{=C_2} \underbrace{- \frac{g}{3}\varepsilon^{\mu\nu\rho\sigma} f^{abc}\partial_\mu\left(A^a_\nu A^b_\rho A^c_\sigma\right)}_{=C_3} = 0. \tag{10}$$

Let us discuss each term separately. First

$$C_1 = -\frac{g}{2}\varepsilon^{\mu\nu\rho\sigma} f^{abc} A^b_\mu A^c_\nu \left(\partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho + g\, f^{ars} A^r_\rho A^s_\sigma\right) \tag{11}$$

$$= -g\varepsilon^{\mu\nu\rho\sigma} f^{abc} \left(\partial_\mu A^a_\nu\right) A^b_\rho A^c_\sigma \tag{12}$$

$$+ \frac{g^2}{2}\varepsilon^{\mu\nu\rho\sigma} f^{abc} f^{ars} A^b_\mu A^c_\nu A^r_\rho A^s_\sigma. \tag{13}$$

In (12) we have used the antisymmetry property of $\varepsilon^{\mu\nu\rho\sigma}$ and we also rearranged the indices appropriately. We will show now that (13) is zero. Indeed

$$\varepsilon^{\mu\nu\rho\sigma} f^{abc} f^{ars} A^b_\mu A^c_\nu A^r_\rho A^s_\sigma = -\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left([\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]\,[\boldsymbol{A}_\rho, \boldsymbol{A}_\sigma]\right) \tag{14}$$

where

$$\boldsymbol{A}_\mu = A^a_\mu T^a. \tag{15}$$

We have

$$\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left([\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]\,[\boldsymbol{A}_\rho, \boldsymbol{A}_\sigma]\right)$$
$$= \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(\boldsymbol{A}_\mu\boldsymbol{A}_\nu\boldsymbol{A}_\rho\boldsymbol{A}_\sigma - \boldsymbol{A}_\mu\boldsymbol{A}_\nu\boldsymbol{A}_\sigma\boldsymbol{A}_\rho - \boldsymbol{A}_\nu\boldsymbol{A}_\mu\boldsymbol{A}_\rho\boldsymbol{A}_\sigma + \boldsymbol{A}_\nu\boldsymbol{A}_\mu\boldsymbol{A}_\sigma\boldsymbol{A}_\rho\right). \tag{16}$$

Each term under the trace can be permuted cyclically, however ε changes the sign

$$\begin{aligned}\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(\boldsymbol{A}_\mu\boldsymbol{A}_\nu\boldsymbol{A}_\rho\boldsymbol{A}_\sigma\right) &= \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(\boldsymbol{A}_\sigma\boldsymbol{A}_\mu\boldsymbol{A}_\nu\boldsymbol{A}_\rho\right)\\ &= -\varepsilon^{\sigma\mu\nu\rho}\,\mathrm{Tr}\left(\boldsymbol{A}_\sigma\boldsymbol{A}_\mu\boldsymbol{A}_\nu\boldsymbol{A}_\rho\right)\\ &= -\varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\left(\boldsymbol{A}_\mu\boldsymbol{A}_\nu\boldsymbol{A}_\rho\boldsymbol{A}_\sigma\right)\\ &= 0.\end{aligned} \tag{17}$$

Hence

$$C_1 = -g\varepsilon^{\mu\nu\rho\sigma} f^{abc} \left(\partial_\mu A^a_\nu\right) A^b_\rho A^c_\sigma. \tag{18}$$

Next

$$
\begin{aligned}
C_2 &= \varepsilon^{\mu\nu\rho\sigma} A^a_\nu \partial_\mu \left(\partial_\rho A^a_\sigma - \partial_\sigma A^a_\rho + g\, f^{ars} A^r_\rho A^s_\sigma \right) \\
&= g \varepsilon^{\mu\nu\rho\sigma} f^{abc} A^a_\nu\, \partial_\mu \left(A^b_\rho A^c_\sigma \right) \\
&= 2 \varepsilon^{\mu\nu\rho\sigma} f^{abc} \left(\partial_\mu A^a_\nu \right) A^b_\rho A^c_\sigma .
\end{aligned} \tag{19}
$$

We have used the fact that antisymmetrized $\partial_\mu \partial_\rho$ vanishes, and then we have rearranged indices to have the same order as in (18).

Finally

$$
\begin{aligned}
C_3 &= -\frac{g}{3} \varepsilon^{\mu\nu\rho\sigma} f^{abc} \partial_\mu \left(A^a_\nu A^b_\rho A^c_\sigma \right) \\
&= -g \varepsilon^{\mu\nu\rho\sigma} f^{abc} \left(\partial_\mu A^a_\nu \right) A^b_\rho A^c_\sigma .
\end{aligned} \tag{20}
$$

We see that indeed

$$
C_1 + C_2 + C_3 = 0.
$$