

QCD lecture 7b

November 22

Formulation of QCD in terms of functional integrals:

- Path integrals in QM
- Functional integrals, Grassmann variables
- Chiral anomaly by Fujikawa
- Atiyah-Singer theorem
- Topology of gauge fields, instantions in QM and in QCD
- Quantization of QED and QCD, ghosts

QM - reminder

Schrödinger eq.

$$i\hbar \frac{\partial \Psi(x, t_b)}{\partial t_b} = H\Psi(x, t_b)$$

propagates solution from $a = (x_a, t_a)$ to $b = (x_b, t_b)$ $\Psi(x, t_b) = e^{-\frac{i}{\hbar}H(t_b-t_a)}\Psi(x, t_a)$
(remember H is an operator)

Define propagator: $K(b, a) = \langle x_b | e^{-\frac{i}{\hbar}H(t_b-t_a)} | x_a \rangle$

recall Dirac notation $\Psi(x) = \langle x | \Psi \rangle$ and plane wave solution $\langle x | p \rangle = N e^{\frac{i}{\hbar}px}$
complex conjugate $\langle p | x \rangle = N e^{-\frac{i}{\hbar}px}$

completeness relation $\sum_p |p\rangle \langle p| = \sum_x |x\rangle \langle x| = 1$

We shall use the following normalization: $\langle p | y \rangle = \sqrt{\frac{1}{2\pi\hbar}} e^{-\frac{i}{\hbar}py}$

Path integral for the propagator

$$K(b, a) = \langle x_b | e^{-\frac{i}{\hbar} H(t_b - t_a)} | x_a \rangle$$

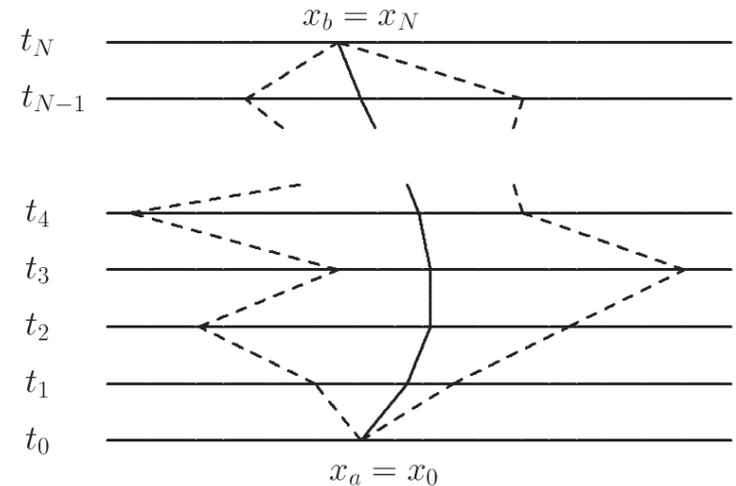
set $\hbar = m = 1$

“slice” evolution operator

$$e^{-i(t_b - t_a)H} = e^{-i\epsilon N H} = e^{-i\epsilon H} e^{-i\epsilon H} \dots e^{-i\epsilon H}$$

insert inbetween unity $1 = \int dx_j |x_j\rangle \langle x_j|$

Discretize time



$$\begin{aligned} \langle x_b | e^{-i(t_b - t_a)H} | x_a \rangle &= \int \langle x_b | e^{-i\epsilon H} | x_{N-1} \rangle dx_{N-1} \langle x_{N-1} | e^{-i\epsilon H} | x_{N-2} \rangle \\ &\dots \langle x_2 | e^{-i\epsilon H} | x_1 \rangle dx_1 \langle x_1 | e^{-i\epsilon H} | x_a \rangle \end{aligned}$$

Path integral for the propagator

Decompose hamiltonian $H = \frac{p^2}{2m} + V(x) = K + V$

and use: $e^{-i\epsilon H} = e^{-i\epsilon(K+V)} = e^{-i\epsilon K} e^{-i\epsilon V} + O(\epsilon^2)$

which is true only for small ϵ

Baker-Cambell-Hausdorff: define C $e^A e^B = e^C$

then $C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] + \frac{1}{12}[[A, B], B] + \dots$
 $\sim \epsilon^2$

Therefore

$$\begin{aligned} K(b, a) &= \langle x_b | e^{-i(t_b - t_a)H} | x_a \rangle \\ &= \int \langle x_b | e^{-i\epsilon K} | x_{N-1} \rangle e^{-i\epsilon V(x_{N-1})} dx_{N-1} \langle x_{N-1} | e^{-i\epsilon K} | x_{N-2} \rangle \\ &\quad \times e^{-i\epsilon V(x_{N-2})} dx_{N-2} \dots dx_1 \langle x_1 | e^{-i\epsilon K} | x_a \rangle e^{-i\epsilon V(x_a)} \end{aligned}$$

Path integral for the propagator

We need to calculate $\langle x | e^{-\frac{i}{\hbar} \epsilon K} | y \rangle = \int dp \langle x | e^{-\frac{i}{\hbar} \epsilon \frac{p^2}{2m}} | p \rangle \langle p | y \rangle$
 (distinguish operators from eigenvalues)

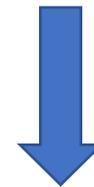
$$= \int dp \langle x | p \rangle e^{-\frac{i \epsilon p^2}{\hbar 2m}} \langle p | y \rangle$$

recall normalization $\langle p | y \rangle = \sqrt{\frac{1}{2\pi\hbar}} e^{-\frac{i}{\hbar} p y}$

$$\langle x | e^{-\frac{i}{\hbar} \epsilon K} | y \rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dp e^{-\frac{i \epsilon p^2}{2m\hbar}} e^{-\frac{i}{\hbar} (y-x)p} = \sqrt{\frac{m}{2i\pi\hbar\epsilon}} e^{im \frac{(y-x)^2}{2\epsilon\hbar}}$$

where we have used

$$\int_{-\infty}^{+\infty} dx e^{ax^2+bx} = \sqrt{\frac{\pi}{-a}} e^{-\frac{b^2}{4a}}, \quad \text{Re } a \leq 0$$



but remember:

$$L_j = \frac{1}{2} m \left(\frac{x_{j+1} - x_j}{\epsilon} \right)^2 - V(x_j)$$

$$\frac{i}{2\hbar} m \left(\frac{y-x}{\epsilon} \right)^2 \epsilon = \frac{i}{\hbar} \frac{mv^2}{2} \epsilon$$

Path integral for the propagator

$$K(b, a) = \lim_{\epsilon \rightarrow 0} \sqrt{\frac{m}{2i\epsilon\hbar\pi}} \int \prod_{j=1}^{N-1} dx_j \sqrt{\frac{m}{2i\epsilon\hbar\pi}} \prod_{k=0}^{N-1} e^{\frac{i}{\hbar}\epsilon L_k}$$

$$\stackrel{\text{def}}{=} \int [\mathcal{D}x(t)] e^{\frac{i}{\hbar} \int_{t_a}^{t_b} dt L[x(t), \dot{x}(t)]}$$

Define functional integration measure
integration over all trajectories from
 a to b

$$[\mathcal{D}x(t)] = dx_1 \dots dx_{N-1} \left(\frac{m}{2i\epsilon\hbar\pi} \right)^{\frac{1}{2}N}$$

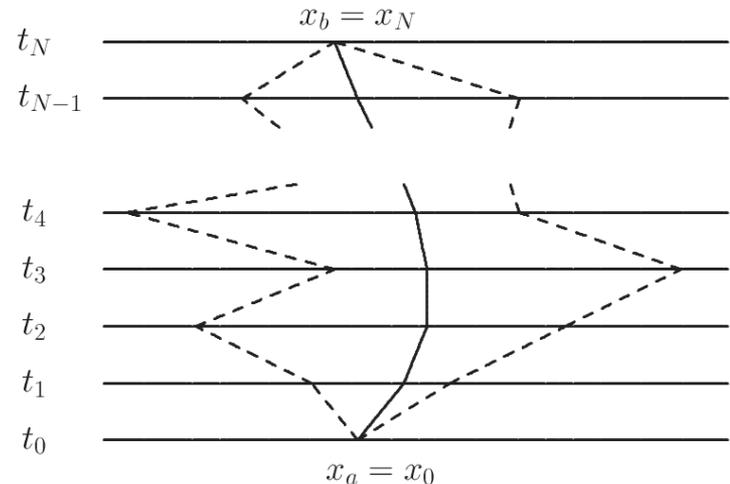
and use definition of action

$$\lim_{\epsilon \rightarrow 0} \sum_{j=0}^{N-1} \epsilon L_j = \int_{t_a}^{t_b} dt L(x(t), \dot{x}(t)) = S[x(t)]$$

to arrive at

$$K(b, a) = \int [\mathcal{D}x(t)] e^{\frac{i}{\hbar} S[x(t)]}$$

special role of the classical trajectory
i.e. stationary point of action



Euclidean path integral

Change $t \rightarrow -i\tau$

then $K(x_b, x_a, -i\tau) = \langle x_b | e^{-\frac{H}{\hbar}\tau} | x_a \rangle$

$$= \sum_{n,n'} \langle x_b | E_n \rangle \langle E_n | e^{-\frac{H}{\hbar}\tau} | E_{n'} \rangle \langle E_{n'} | x_a \rangle$$

$$= \sum_n e^{-\frac{E_n}{\hbar}\tau} \phi_n(x_b) \phi_n^*(x_a).$$

for large τ only the ground state survives

Feynman-Kac formula $E_0 = - \lim_{\tau \rightarrow \infty} \left\{ \frac{\hbar}{\tau} \ln (K(x_b, x_a, -i\tau)) \right\}$

In Euclidean one can perform computer simulations

$$K_n(x, x, -i\tau) = \int dx_1 dx_2 \dots dx_n \left(\frac{1}{2\pi\epsilon} \right)^{\frac{1}{2}(n+1)} e^{-\epsilon \sum_{j=0}^n \left\{ \frac{1}{2} \left(\frac{x_{j+1} - x_j}{\epsilon} \right)^2 + V(x_j) \right\}}$$

Gaussian functional integrals

Assume that path integral is the way we formulate QM (and QFT). All properties and equations are derived from the path integral. In practice we deal with Gaussian functional integrals:

$$L(\dot{x}, x, t) = a(t) \dot{x}^2(t) + b(t) \dot{x}x + c(t) x^2 + d(t)\dot{x} + e(t)x + f(t)$$

Propagator:

$$K(x_b, x_a, t_b - t_a) = \int [\mathcal{D}x(t)] e^{\frac{i}{\hbar} \int_{t_a}^{t_b} dt L(\dot{x}, x, t)}$$

To evaluate K decompose the quantal trajectory into the classical one $\bar{x}(t)$

$$\delta S[x(t)] = 0 \quad \text{gives} \quad \bar{x}(t)$$

and a fluctuation $y(t)$:

$$x(t) = \bar{x}(t) + y(t), \quad y(t_b) = y(t_a) = 0$$

Since terms linear in y vanish $S[\bar{x}(t) + y(t)] = S[\bar{x}(t)] + \frac{1}{2} \delta^2 S[y(t)]$

for convenience $T = t_b - t_a$

$$= S[\bar{x}] + \frac{1}{2!} \int_0^T \left[\frac{\partial^2 L}{\partial \dot{x}^2} \dot{y}^2 + 2 \frac{\partial^2 L}{\partial x \partial \dot{x}} \dot{y}y + \frac{\partial^2 L}{\partial x^2} y^2 \right] dt$$

Gaussian functional integrals

Since $\bar{x}(t)$ is fixed we have $\mathcal{D}x(t) = \mathcal{D}y(t)$
and

$$K(x_b, x_a, t_b - t_a) = F(t_b - t_a) e^{\frac{i}{\hbar} S[\bar{x}(t)]}$$

where

$$F(t_b - t_a) = \int [\mathcal{D}y(t)] e^{\frac{i}{\hbar} \frac{1}{2} \delta^2 S[y(t)]}$$

Recall:

$$\delta^2 S = \int_0^T \left[\dot{y} \frac{\partial^2 L}{\partial \dot{x}^2} \dot{y} + 2y \frac{\partial^2 L}{\partial x \partial \dot{x}} \dot{y} + y \frac{\partial^2 L}{\partial x^2} y \right] dt$$

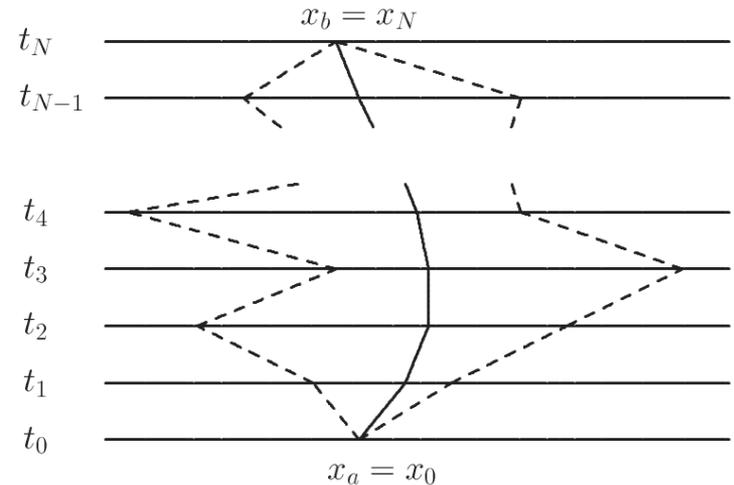
identities:

(integration by parts) $\dot{y} \frac{\partial^2 L}{\partial \dot{x}^2} \dot{y} = \frac{d}{dt} \left(y \frac{\partial^2 L}{\partial \dot{x}^2} \dot{y} \right) - y \frac{d}{dt} \left(\frac{\partial^2 L}{\partial \dot{x}^2} \dot{y} \right)$

$$y(0) = y(T) = 0$$

$$2y \frac{\partial^2 L}{\partial x \partial \dot{x}} \dot{y} = \frac{d}{dt} \left(\frac{\partial^2 L}{\partial x \partial \dot{x}} y^2 \right) - y \frac{d}{dt} \left(\frac{\partial^2 L}{\partial x \partial \dot{x}} \right) y$$

we get $\delta^2 S = - \int_0^T dt y \left[\frac{d}{dt} \left(\frac{\partial^2 L}{\partial \dot{x}^2} \frac{d}{dt} \right) + \left(\frac{d}{dt} \frac{\partial^2 L}{\partial x \partial \dot{x}} \right) - \frac{\partial^2 L}{\partial x^2} \right] y = \int_0^T dt y D(t) y$
definition of D



Gaussian functional integrals

$$\delta^2 S = - \int_0^T dt y \left[\frac{d}{dt} \left(\frac{\partial^2 L}{\partial \dot{x}^2} \frac{d}{dt} \right) + \left(\frac{d}{dt} \frac{\partial^2 L}{\partial x \partial \dot{x}} \right) - \frac{\partial^2 L}{\partial x^2} \right] y = \int_0^T dt y D(t) y$$

D is a Sturm-Liouville operator $D(t)y_n(t) = \lambda_n y_n(t)$, $n = 1, 2, 3, \dots$, $\lambda_1 < \lambda_2 < \dots$

Example: $L = \frac{1}{2}m\dot{x}^2 - V(x)$ $D(t) = -m \frac{\partial^2}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} \Big|_{x=\bar{x}(t)}$

Use y_n basis to expand $y(t) = \sum_{n=1}^{\infty} a_n y_n(t)$ **then** $\delta^2 S[y] = \sum_{n=1}^{\infty} \lambda_n a_n^2$

and $[Dy(t)] \sim \prod_{n=1}^{\infty} da_n$

$$F(T) \sim \prod_{n=1}^{\infty} da_n \exp \left(\frac{i}{2\hbar} \lambda_n a_n^2 \right) \sim \sqrt{\frac{1}{\prod_n \lambda_n}} = \sqrt{\frac{1}{\det D(t)}}$$

Path integral revisited

We have performed dp integral using a specific form of the hamiltonian

$$H = \frac{p^2}{2m} + V(x)$$

however we do need to use this information. We only have to remember

$$\mathcal{L} = p\dot{q} - H(p, q)$$

Let's recalculate

$$\begin{aligned}\langle x, t + \varepsilon | y, t \rangle &= \langle x | e^{-iH\varepsilon} | y \rangle \\ &= \int \frac{dp}{2\pi} e^{ip(x-y)} e^{-iH\varepsilon} \\ &= \int \frac{dp}{2\pi} \exp i \left[p \frac{(x-y)}{\varepsilon} - H(p, x) \right] \varepsilon \\ &= \int \frac{dp}{2\pi} \exp i [p\dot{x} - H(p, x)] \varepsilon\end{aligned}$$

Hence:

$$K(b, a) \sim \int \mathcal{D}[x(t)] \int \mathcal{D}[p(t)] \exp \left(\frac{i}{\hbar} \int dt [p\dot{x} - H(p, x)] \right)$$

Transition amplitudes

Consider matrix element of a position operator Q measuring expectation value of the position at time t_I

$$\langle q_f | e^{-i\mathcal{H}(t_f-t_1)} Q e^{-i\mathcal{H}(t_1-t_i)} | q_i \rangle$$

We have

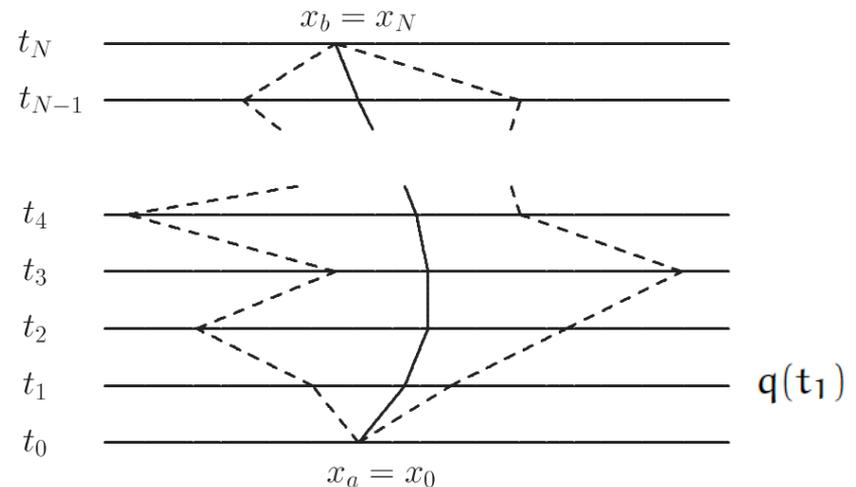
$$Q \rightarrow \int dq dq' |q\rangle \underbrace{\langle q|Q|q'\rangle}_{q \delta(q-q')} \langle q'| = \int dq q |q\rangle \langle q|$$

which lead to

$$\langle q_f | e^{-i\mathcal{H}(t_f-t_1)} Q e^{-i\mathcal{H}(t_1-t_i)} | q_i \rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)] q(t_1) e^{i\mathcal{S}[q(t)]}$$

Similarly for $t_2 > t_1$

$$\begin{aligned} \langle q_f | e^{-i\mathcal{H}(t_f-t_2)} Q e^{-i\mathcal{H}(t_2-t_1)} Q e^{-i\mathcal{H}(t_1-t_i)} | q_i \rangle &= \\ &= \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)] q(t_1) q(t_2) e^{i\mathcal{S}[q(t)]} \end{aligned}$$



Transition amplitudes

F. Gelis: A Stroll Through Field Theory

Define time dependent operator $Q(t) \equiv e^{i\mathcal{H}t} Q e^{-i\mathcal{H}t}$ and $|q, t\rangle \equiv e^{i\mathcal{H}t} |q\rangle$

then

$$\langle q_f, t_f | Q(t_2) Q(t_1) | q_i, t_i \rangle \underset{t_2 > t_1}{=} \int_{\substack{q(t_i) = q_i \\ q(t_f) = q_f}} [Dq(t)] q(t_1) q(t_2) e^{iS[q(t)]}$$

operators functions

Note that l.h.s is very different when $t_1 > t_2$, whereas r.h.s. is the same because classical trajectories commute. Introduce time ordering T

$$T(Q(t_1)Q(t_2)) = \theta(t_1 - t_2)Q(t_1)Q(t_2) + \theta(t_2 - t_1)Q(t_2)Q(t_1)$$

then

$$\langle q_f, t_f | T(Q(t_1)Q(t_2)) | q_i, t_i \rangle = \int_{\substack{q(t_i) = q_i \\ q(t_f) = q_f}} [Dq(t)] q(t_1) q(t_2) e^{iS[q(t)]}$$

generally

$$\langle q_f, t_f | T(Q(t_1) \cdots Q(t_n)) | q_i, t_i \rangle = \int_{\substack{q(t_i) = q_i \\ q(t_f) = q_f}} [Dq(t)] q(t_1) \cdots q(t_n) e^{iS[q(t)]}$$

Functional sources and derivatives

One can derive transition amplitudes with the help of generating functional

$$Z_{fi}[j(t)] \equiv \langle q_f, t_f | T \exp i \int_{t_i}^{t_f} dt j(t) Q(t) | q_i, t_i \rangle$$

where $j(t)$ is some arbitrary function of time and $Q(t)$ is an operator
Amplitudes are given as functional derivatives

$$\langle q_f, t_f | T (Q(t_1) \cdots Q(t_n)) | q_i, t_i \rangle = \frac{\delta^n Z_{fi}[j]}{i^n \delta j(t_1) \cdots \delta j(t_n)} \Big|_{j=0}$$

Functional derivatives act essentially as regular differentiation with one additional property

$$\frac{\delta j(t)}{\delta j(t')} = \delta(t - t') \quad \text{values of function } j(t) \text{ at different times are independent variables}$$

Generating functional has path integral representation (Lagrange)

$$Z_{fi}[j(t)] = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)] e^{iS[q(t)] + i \int_{t_i}^{t_f} dt j(t) q(t)}$$

Functional sources and derivatives

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Functional derivatives act essentially as regular differentiation with one additional property

$$\frac{\delta j(t)}{\delta j(t')} = \delta(t - t') \quad \text{values of function } j(t) \text{ at different times are independent variables}$$

Generating functional has path integral representation (Hamilton)

$$Z_{fi}[j(t)] = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dp(t) Dq(t)] \times \exp \left\{ i \int_{t_i}^{t_f} dt (p(t) \dot{q}(t) - \mathcal{H}(p(t), q(t)) + j(t) q(t)) \right\}$$

Ground state projection

Initial and final states do not have to be position eigenstates. Consider some operator O and some state ψ

$$\psi(q) \equiv \langle q | \psi \rangle$$

Then

$$\langle \psi_f, t_f | O | \psi_i, t_i \rangle = \int dq_i dq_f \psi_f^*(q_f) \psi_i(q_i) \langle q_f, t_f | O | q_i, t_i \rangle$$

In practice we often need matrix element when initial and final states are the ground states:

$$\begin{aligned} |q_i, t_i\rangle &= e^{i\mathcal{H}t_i} |q_i\rangle \\ &= \sum_{n=0}^{\infty} e^{i\mathcal{H}t_i} |n\rangle \langle n | q_i \rangle \\ &= \sum_{n=0}^{\infty} \psi_n^*(q_i) e^{iE_n t_i} |n\rangle \end{aligned}$$

Assume that $E_0 = 0$ (shifting energy) and multiply the hamiltonian by $1 - i0^+$

Then all factors $\exp(i(1 - i0^+)E_n t_i)$ go to 0 for $t_i \rightarrow -\infty$ except for the ground state

Ground state projection

With $1 - i0^+$ prescripton

$$\lim_{t_i \rightarrow -\infty} |q_i, t_i\rangle = \psi_0^*(q_i) |0\rangle \quad \lim_{t_f \rightarrow +\infty} \langle q_f, t_f| = \psi_0(q_f) \langle 0|$$

The generating functional is then vacuum expectation value and reads (Hamilton)

$$Z[j(t)] = \int [Dp(t)Dq(t)] \\ \times \exp \left\{ i \int dt \left(p(t)\dot{q}(t) - \underline{(1 - i0^+)} \mathcal{H}(p(t), q(t)) + j(t)q(t) \right) \right\}$$

or (Lagrange)

$$Z[j(t)] = \int [Dq(t)] \\ \times \exp \left\{ i \int dt \left((1 + i0^+) \frac{m\dot{q}^2(t)}{2} - (1 - i0^+) V(q(t)) + j(t)q(t) \right) \right\}$$

Normalization $Z[0] = 1$