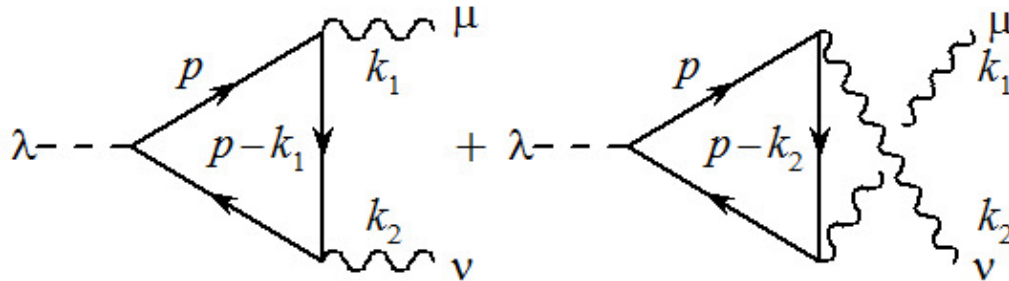# QCD lecture 7a

November 22

Axial current conservation

$$k_1^\mu T_{\mu\nu\lambda} = \Delta_{\nu\lambda},$$
$$q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}$$

Axial current conservation

$$p \to p + k_1$$

$$\Delta_{\nu\lambda} = \int \frac{d^4 p}{(2\pi)^4} \operatorname{Tr} \left[\gamma_\lambda \gamma_5 \frac{i}{(\not{p} - \not{q}) - m} \gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m} - \gamma_\lambda \gamma_5 \frac{i}{(\not{p} - \not{k}_2) - m} \gamma_\nu \frac{i}{\not{p} - m} \right]$$

$$\Delta_{\mu\nu}^{(1)} = \int \frac{d^4 p}{(2\pi)^4} \operatorname{Tr} \left[\frac{i}{\not{p} - m} \gamma_5 \gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m} \gamma_\mu - \frac{i}{(\not{p} - \not{k}_2) - m} \gamma_5 \gamma_\nu \frac{i}{(\not{p} - \not{q}) - m} \gamma_\mu \right]$$

$$\Delta_{\mu\nu}^{(2)} = \int \frac{d^4 p}{(2\pi)^4} \operatorname{Tr} \left[\frac{i}{\not{p} - m} \gamma_5 \gamma_\mu \frac{i}{(\not{p} - \not{k}_2) - m} \gamma_\nu - \frac{i}{(\not{p} - \not{k}_1) - m} \gamma_5 \gamma_\mu \frac{i}{(\not{p} - \not{q}) - m} \gamma_\nu \right]$$

It seems that by changing integration variables one can nullify all three Deltas. However, because all these integrals are linearly divergent, change of variables may result in a finite value for the difference of two infinities.

$$p \to p + k_2$$

$$p \to p + k_1$$

Mathematical diggression

Consider the integral that is naively zero:

$$\int_{-\infty}^{\infty} dx \left[f(x + a) - f(x) \right]$$

However, if

$$f(\pm\infty) \neq 0.$$

we can calculate this integral by Taylor expansion:

$$\int_{-\infty}^{\infty} dx \left[f(x + a) - f(x) \right] = a \left[f(\infty) - f(-\infty) \right] + \frac{a^2}{2} \left[f'(\infty) - f'(-\infty) \right] + \ldots$$

it may happen that $\neq 0$

Mathematical diggression

Consider Euclidean integral:

$$\Delta(\vec{a}) = \int d^n\vec{r}\,[f(\vec{r}+\vec{a}) - f(\vec{r})]$$

expand in a

$$= \int d^n\vec{r}\,\vec{a}\cdot\vec{\nabla}f(\vec{r}) + \dots$$

apply Gauss theorem

$$= \vec{a}\cdot\vec{n}\,S_n(R)\,f(\vec{R})$$

where $\vec{n} = \dfrac{\vec{R}}{R}$ and $S_n(R)$ is a surface of the n sphere, R is regulator.

For even n

$$S_n(R) = \frac{2\pi^{n/2}}{(n/2-1)!}R^{n-1} = \begin{cases} 2\pi R & \text{for} \quad n = 2 \\[2mm] 2\pi^2 R^3 & \text{for} \quad n = 4 \end{cases}$$

In Minkowski space

$$\Delta(a) = 2i\pi^2 a^\mu \lim_{R\to\infty} R^2 R_\mu\, f(R)$$

Shift in full amplitude

$$T_{\mu\nu\lambda} = -i \int \frac{d^4 p}{(2\pi)^4} \, \mathrm{Tr} \left[\frac{i}{\not{p} - m} \gamma_\lambda \gamma_5 \frac{i}{(\not{p} - \not{q}) - m} \gamma_\nu \frac{i}{(\not{p} - \not{k_1}) - m} \gamma_\mu \right]$$

$$-i \int \frac{d^4 p}{(2\pi)^4} \, \mathrm{Tr} \left[\frac{i}{\not{p} - m} \gamma_\lambda \gamma_5 \frac{i}{(\not{p} - \not{q}) - m} \gamma_\mu \frac{i}{(\not{p} - \not{k_2}) - m} \gamma_\nu \right]$$

define shift vector
and amplitude difference:

$$a = \alpha k_1 + (\alpha - \beta) k_2$$

$$\Delta_{\mu\nu\lambda}(a) = T_{\mu\nu\lambda}(p \to p + a) - T_{\mu\nu\lambda}$$

Strategy:

$$q^\lambda T_{\mu\nu l}(a) = q^\lambda \left(T_{\mu\nu l}(a) - T_{\mu\nu l}(0) \right) + q^\lambda T_{\mu\nu l}(0)$$
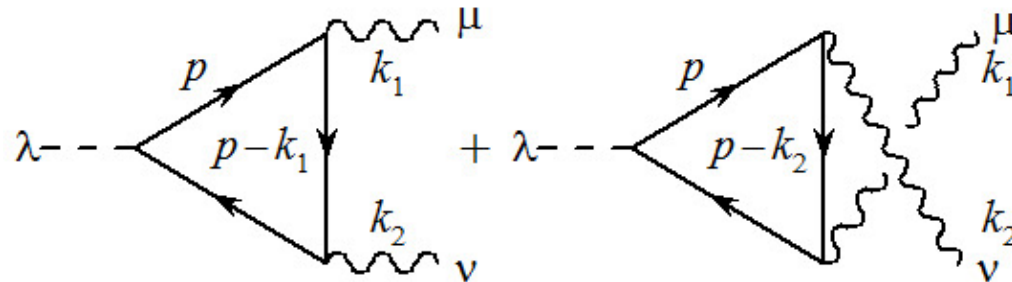$$= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2m T_{\mu\nu} + \Delta_{\mu\nu}^{(1)} + \Delta_{\mu\nu}^{(2)}$$

$$k_1^\mu T_{\mu\nu l}(a) = k_1^\mu \left(T_{\mu\nu l}(a) - T_{\mu\nu l}(0) \right) + k_1^\mu T_{\mu\nu l}(0)$$

chose a in a way that vector current is conserved
and see what comes out for the axial current

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \left\{ \mathrm{Tr}\left[\frac{1}{\not{p}+\not{a}-m}\gamma_\lambda\gamma_5 \frac{1}{(\not{p}+\not{a}-\not{q})-m}\gamma_\nu \frac{1}{(\not{p}+\not{a}-\not{k}_1)-m}\gamma_\mu \right] \right.$$

$$\left. - \mathrm{Tr}\left[\frac{1}{\not{p}-m}\gamma_\lambda\gamma_5 \frac{1}{(\not{p}-\not{q})-m}\gamma_\nu \frac{1}{(\not{p}-\not{k}_1)-m}\gamma_\mu \right] \right\}$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2) .$$

Shift in full amplitude

Calculate
(all *i*'s give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \left\{ \text{Tr} \left[\frac{1}{\not{p}+\not{a}-m} \gamma_\lambda \gamma_5 \frac{1}{(\not{p}+\not{a}-\not{q})-m} \gamma_\nu \frac{1}{(\not{p}+\not{a}-\not{k}_1)-m} \gamma_\mu \right] \right.$$

$$\left. - \text{Tr} \left[\frac{1}{\not{p}-m} \gamma_\lambda \gamma_5 \frac{1}{(\not{p}-\not{q})-m} \gamma_\nu \frac{1}{(\not{p}-\not{k}_1)-m} \gamma_\mu \right] \right\}$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2).$$

Expand in *a*

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} \text{Tr} \left[\frac{1}{\not{p}-m} \gamma_\lambda \gamma_5 \frac{1}{(\not{p}-\not{q})-m} \gamma_\nu \frac{1}{(\not{p}-\not{k}_1)-m} \gamma_\mu \right]$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2).$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \left\{ \text{Tr}\left[\frac{1}{\not{p} + \not{a} - m} \gamma_\lambda \gamma_5 \frac{1}{(\not{p} + \not{a} - \not{q}) - m} \gamma_\nu \frac{1}{(\not{p} + \not{a} - \not{k}_1) - m} \gamma_\mu \right] \right.$$

$$\left. - \text{Tr}\left[\frac{1}{\not{p} - m} \gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{q}) - m} \gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m} \gamma_\mu \right] \right\}$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2).$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} \text{Tr}\left[\frac{1}{\not{p} - m} \gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{q}) - m} \gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m} \gamma_\mu \right]$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2).$$

large p limit

$$\frac{1}{p^6} \text{Tr}\left[\not{p} \gamma_\lambda \gamma_5 \not{p} \gamma_\nu \not{p} \gamma_\mu \right]$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \left\{ \text{Tr}\left[\frac{1}{\not{p}+\not{a}-m}\gamma_\lambda\gamma_5 \frac{1}{(\not{p}+\not{a}-\not{q})-m}\gamma_\nu \frac{1}{(\not{p}+\not{a}-\not{k}_1)-m}\gamma_\mu \right] \right.$$

$$\left. - \text{Tr}\left[\frac{1}{\not{p}-m}\gamma_\lambda\gamma_5 \frac{1}{(\not{p}-\not{q})-m}\gamma_\nu \frac{1}{(\not{p}-\not{k}_1)-m}\gamma_\mu \right] \right\}$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2).$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} \text{Tr}\left[\frac{1}{\not{p}-m}\gamma_\lambda\gamma_5 \frac{1}{(\not{p}-\not{q})-m}\gamma_\nu \frac{1}{(\not{p}-\not{k}_1)-m}\gamma_\mu \right]$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2).$$

large p limit

$$\frac{1}{p^6}\text{Tr}\left[\not{p}\gamma_\lambda\gamma_5\not{p}\gamma_\nu\not{p}\gamma_\mu \right]$$

apply expansion

$$\Delta_{\mu\nu\lambda}(a) = -\frac{i}{(2\pi)^4}2\pi^2 a^\sigma \lim_{P\to\infty} P^3 \frac{P_\sigma}{P} \text{Tr}\left[\not{P}\gamma_\lambda\gamma_5\not{P}\gamma_\nu\not{P}\gamma_\mu \right]\frac{1}{P^6}$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

$$\Delta(a) = 2i\pi^2 a^\mu \lim_{R\to\infty} R^2 R_\mu \ f(R)$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \left\{ \mathrm{Tr}\left[\frac{1}{\slashed{p}+\slashed{a}-m}\gamma_\lambda\gamma_5 \frac{1}{(\slashed{p}+\slashed{a}-\slashed{q})-m}\gamma_\nu \frac{1}{(\slashed{p}+\slashed{a}-\slashed{k}_1)-m}\gamma_\mu \right] \right.$$

$$\left. - \mathrm{Tr}\left[\frac{1}{\slashed{p}-m}\gamma_\lambda\gamma_5 \frac{1}{(\slashed{p}-\slashed{q})-m}\gamma_\nu \frac{1}{(\slashed{p}-\slashed{k}_1)-m}\gamma_\mu \right] \right\}$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2).$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} \mathrm{Tr}\left[\frac{1}{\slashed{p}-m}\gamma_\lambda\gamma_5 \frac{1}{(\slashed{p}-\slashed{q})-m}\gamma_\nu \frac{1}{(\slashed{p}-\slashed{k}_1)-m}\gamma_\mu \right]$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2).$$

large p limit

$$\frac{1}{p^6}\mathrm{Tr}\left[\slashed{p}\gamma_\lambda\gamma_5\slashed{p}\gamma_\nu\slashed{p}\gamma_\mu\right]$$

apply expansion

$$\Delta_{\mu\nu\lambda}(a) = -\frac{i}{(2\pi)^4} 2\pi^2 a^\sigma \lim_{P\to\infty} P^3 \frac{P_\sigma}{P} \mathrm{Tr}\left[\slashed{P}\gamma_\lambda\gamma_5\slashed{P}\gamma_\nu\slashed{P}\gamma_\mu\right] \frac{1}{P^6}$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

calculate Trace

$$\mathrm{Tr}\left[\slashed{P}\gamma_\lambda\gamma_5\slashed{P}\gamma_\nu\slashed{P}\gamma_\mu\right] = 4iP^2\varepsilon_{\alpha\mu\nu\lambda}P^\alpha$$

Remember that $\varepsilon_{\alpha\mu\nu\lambda} = -\varepsilon^{\alpha\mu\nu\lambda}$

Shift in full amplitude

We arrive at:

$$\Delta_{\mu\nu\lambda}(a) = \frac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha} \, a_\sigma \lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

Shift in full amplitude

We arrive at:
$$\Delta_{\mu\nu\lambda}(a) = \frac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha} \, a_\sigma \lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

take symmetric limit:
$$\lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} = \frac{1}{4} g^{\sigma\alpha}$$

recall:
$$a = \alpha k_1 + (\alpha - \beta)k_2$$

Shift in full amplitude

We arrive at: $\Delta_{\mu\nu\lambda}(a) = \dfrac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha}\, a_\sigma \lim_{P\to\infty} \dfrac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu,\, k_1 \leftrightarrow k_2)$

take symmetric limit:
$$\lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} = \frac{1}{4} g^{\sigma\alpha}$$

recall: $a = \alpha k_1 + (\alpha - \beta)k_2$

Final result:
$$\begin{aligned}
\Delta_{\mu\nu\lambda}(a) &= \frac{1}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} a^\alpha + (\mu \longleftrightarrow \nu,\, k_1 \leftrightarrow k_2) \\
&= \frac{1}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} (\alpha k_1^\alpha + (\alpha - \beta)k_2^\alpha - \alpha k_2^\alpha - (\alpha - \beta)k_1^\alpha) \\
&= \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} (k_1 - k_2)^\alpha \,.
\end{aligned}$$

depends on β, there is an ambiguity, which we have to fix demanding that vector current is conserved.

Axial current, cont.

Recall:

$$
\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0) \right) + q^\lambda T_{\mu\nu\lambda}(0) \\
&= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2m T_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}
\end{aligned}
$$

calculated   finite   needs to be computed

Axial current, cont.

Recall:

$$q^\lambda T_{\mu\nu\lambda}(a) = q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0) \right) + q^\lambda T_{\mu\nu\lambda}(0)$$

$$= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2m T_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}$$

calculated   finite   needs to be computed

Let's calculate

$$\Delta^{(1)}_{\mu\nu} = \int \frac{d^4 p}{(2\pi)^4} \, \mathrm{Tr} \left[\frac{i}{\not{p} - m} \gamma_5 \gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m} \gamma_\mu - \frac{i}{(\not{p} - \not{k}_2) - m} \gamma_5 \gamma_\nu \frac{i}{(\not{p} - \not{q}) - m} \gamma_\mu \right]$$

Axial current, cont.

Recall:
$$q^\lambda T_{\mu\nu\lambda}(a) = q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0)$$
$$= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2m T_{\mu\nu} + \Delta_{\mu\nu}^{(1)} + \Delta_{\mu\nu}^{(2)}$$

calculated   finite   needs to be computed

Let's calculate (multiply i's and change order)

$$\Delta_{\mu\nu}^{(1)} = \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\frac{i}{\not{p} - m} \gamma_5 \gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m} \gamma_\mu - \frac{i}{(\not{p} - \not{k}_2) - m} \gamma_5 \gamma_\nu \frac{i}{(\not{p} - \not{q}) - m} \gamma_\mu \right]$$

$$= \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left[\frac{1}{(\not{p} - \not{k}_2) - m} \gamma_5 \gamma_\nu \frac{1}{(\not{p} - \not{k}_2 - \not{k}_1) - m} \gamma_\mu - \frac{1}{\not{p} - m} \gamma_5 \gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m} \gamma_\mu \right]$$

We can use the same trick as previously $\quad p \to p - k_2 \quad$ where $\quad a = -k_2$

Axial current, cont.

Recall:
$$q^\lambda T_{\mu\nu\lambda}(a) = q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0)$$
$$= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2m T_{\mu\nu} + \Delta_{\mu\nu}^{(1)} + \Delta_{\mu\nu}^{(2)}$$

calculated   finite   needs to be computed

Let's calculate

$$\Delta_{\mu\nu}^{(1)} = \int \frac{d^4p}{(2\pi)^4} \, \mathrm{Tr}\left[\frac{i}{\not{p} - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m}\gamma_\mu - \frac{i}{(\not{p} - \not{k}_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{q}) - m}\gamma_\mu\right]$$

$$= \int \frac{d^4p}{(2\pi)^4} \, \mathrm{Tr}\left[\frac{1}{(\not{p} - \not{k}_2) - m}\gamma_5\gamma_\nu \frac{1}{(\not{p} - \not{k}_2 - \not{k}_1) - m}\gamma_\mu - \frac{1}{\not{p} - m}\gamma_5\gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m}\gamma_\mu\right]$$

We can use the same trick as previously $\quad p \to p - k_2 \quad$ where $\quad a = -k_2$

$$\Delta_{\mu\nu}^{(1)} = -\frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho \lim_{P \to \infty} \frac{P_\rho}{P^2} \, \mathrm{Tr}\left[\not{P}\gamma_5\gamma_\nu(\not{P} - \not{k}_1)\gamma_\mu\right] \quad$$ keep k_1, because Tr with 2 P's is zero

Axial current, cont.

$$\Delta^{(1)}_{\mu\nu} = -\frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho \lim_{P\to\infty} \frac{P_\rho}{P^2} \operatorname{Tr}\left[\not{P}\gamma_5\gamma_\nu(\not{P}-\not{k}_1)\gamma_\mu\right]$$

We have

$$
\begin{aligned}
\Delta^{(1)}_{\mu\nu} &= \frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho k_1^\sigma \lim_{P\to\infty} \frac{P_\rho P^\alpha}{P^2} \operatorname{Tr}\left[\gamma_\alpha\gamma_5\gamma_\nu\gamma_\sigma\gamma_\mu\right] \\
&= \frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho k_1^\sigma \frac{1}{4}(-)\underbrace{\operatorname{Tr}\left[\gamma_5\gamma_\rho\gamma_\nu\gamma_\sigma\gamma_\mu\right]}_{4i\varepsilon_{\rho\nu\sigma\mu}} \\
&= -\frac{1}{8\pi^2}\varepsilon_{\mu\nu\sigma\rho} k_1^\sigma k_2^\rho.
\end{aligned}
$$

We obtain $\Delta^{(2)}_{\mu\nu}$ by $\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2$, hence

$$\Delta^{(1)}_{\mu\nu} = \Delta^{(2)}_{\mu\nu}$$

Axial current, final

$$
\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0) \right) + q^\lambda T_{\mu\nu\lambda}(0) \\
&= 2m T_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} + q^\lambda \Delta_{\mu\nu\lambda}(a) \\
&= 2m T_{\mu\nu} - \frac{1}{4\pi^2} \varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho + (k_1 + k_2)^\lambda \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} (k_1 - k_2)^\alpha \\
&= 2m T_{\mu\nu} - \frac{1-\beta}{4\pi^2} \varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho
\end{aligned}
$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$\Delta_{\mu\nu\lambda}(a) \quad \downarrow$$

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} (k_1 - k_2)^\alpha \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho.
\end{aligned}
$$

We ned the first piece

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0) \right) + k_1^\mu T_{\mu\nu\lambda}(0) \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} (k_1 - k_2)^\alpha \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}
$$

We ned the first piece

$$
k_1^\mu T_{\mu\nu\lambda} = -\int \frac{d^4 p}{(2\pi)^4}
$$

$$
\left\{ \mathrm{Tr} \left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{q}) - m} \gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m} \right] - \mathrm{Tr} \left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{k}_2) - m} \gamma_\nu \frac{1}{\not{p} - m} \right] \right\}
$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0) \right) + k_1^\mu T_{\mu\nu\lambda}(0) \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} (k_1 - k_2)^\alpha \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}
$$

We ned the first piece

$$
k_1^\mu T_{\mu\nu\lambda} = -\int \frac{d^4 p}{(2\pi)^4}
$$

$$
\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not q) - m} \gamma_\nu \frac{1}{(\not p - \not k_1) - m} \right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2) - m} \gamma_\nu \frac{1}{\not p - m} \right] \right\}
$$

$$
\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2 - \not k_1)) - m} \gamma_\nu \frac{1}{(\not p - \not k_1) - m} \right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2) - m} \gamma_\nu \frac{1}{\not p - m} \right] \right\}
$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0) \right) + k_1^\mu T_{\mu\nu\lambda}(0) \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} (k_1 - k_2)^\alpha \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho.
\end{aligned}
$$

We ned the first piece

$$
k_1^\mu T_{\mu\nu\lambda} = -\int \frac{d^4 p}{(2\pi)^4}
$$

$$
\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{q}) - m} \gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m} \right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{k}_2) - m} \gamma_\nu \frac{1}{\not{p} - m} \right] \right\}
$$

$$
\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{k}_2 - \not{k}_1)) - m} \gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m} \right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{k}_2) - m} \gamma_\nu \frac{1}{\not{p} - m} \right] \right\}
$$

$$
k_1^\mu T_{\mu\nu\lambda} = -\frac{1}{(2\pi)^4} 2i\pi^2 (-) k_1^\sigma \lim_{P\to\infty} \frac{P_\sigma}{P^2} \mathrm{Tr}\left[\gamma_\lambda \gamma_5 (\not{P} - \not{k}_2) \gamma_\nu \not{P} \right]
$$

Vector current

$$k_1^\mu T_{\mu\nu\lambda} = -\frac{1}{(2\pi)^4} 2i\pi^2 (-)k_1^\sigma \lim_{P\to\infty} \frac{P_\sigma}{P^2} \mathrm{Tr}\left[\gamma_\lambda\gamma_5(\slashed{P} - \slashed{k}_2)\gamma_\nu\slashed{P}\right]$$

$$= -\frac{1}{8\pi^2} i\frac{1}{4} \mathrm{Tr}\left[\gamma_\lambda\gamma_5\gamma_\rho\gamma_\nu\gamma_\sigma\right] k_1^\sigma k_2^\rho$$

$$= \frac{1}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho.$$

Recall

$$k_1^\mu T_{\mu\nu\lambda}(a) = k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho}k_1^\sigma k_2^\rho = \frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho}k_1^\sigma k_2^\rho$$

We need to choose $\beta = -1$ to have vector current conserved!

Axial anomaly

Summarizing:

$$q^\lambda T_{\mu\nu\lambda}(a) = 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}k_1^\sigma k_2^\rho$$

$$k_1^\mu T_{\mu\nu\lambda}(a) = \frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho}k_1^\sigma k_2^\rho$$

Choose $\beta = -1$

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} - \frac{1}{2\pi^2}\varepsilon_{\mu\nu\sigma\rho}k_1^\sigma k_2^\rho$$

Axial current is anomalous

This can be translated to the configurations space

$$\partial^\lambda J_\lambda^5(x) = \frac{1}{(4\pi)^2}\varepsilon_{\mu\nu\sigma\rho}F^{\mu\nu}(x)F^{\sigma\rho}(x) + \mathcal{O}(m)$$

Axial anomaly

Summarizing:

$$q^\lambda T_{\mu\nu\lambda}(a) = 2m T_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho$$

$$k_1^\mu T_{\mu\nu\lambda}(a) = \frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho.$$

Choose $\beta = -1$

$$q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} - \frac{1}{2\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho$$

- Anomaly is mass independent
- Adler-Bardeen theorem (69): no higher order correctoons
- name: Adler-Bardeen-Jackiw anomaly
- Fujikawa (79) path integral formulation
- In non-Abelian case one can nullify anomaly Tr(…)=0

Axial current is anomalous

This can be translated to the configurations space

$$\partial^\lambda J_\lambda^5(x) = \frac{1}{(4\pi)^2}\varepsilon_{\mu\nu\sigma\rho} F^{\mu\nu}(x) F^{\sigma\rho}(x) + \mathcal{O}(m)$$