QCD lecture 6a

November 15

DGLAP Evolution Equations

Full set of DGLAP equations:

$$Q^2 \frac{d}{dQ^2} q_i(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{qq} \otimes q_i(Q^2) + P_{qG} \otimes G(Q^2) \right]$$

$$Q^2 \frac{d}{dQ^2} G(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{Gq} \otimes \sum_i q_i(Q^2) + P_{GG} \otimes G(Q^2) \right]$$

We need an input at one scale Q_0^2 and then we can evolve them up to some other Q^2
note that index i runs over quarks and antiquarks
when we construct a difference, called non-singlet, gluons cancel

$$q_i^{NS}(x, Q^2) = q_i(x, Q^2) - \overline{q}_i(x, Q^2)$$

DGLAP Evolution Equations

Define:

singlet

$$q^S(x, Q^2) = \sum_i \left(q_i(x, Q^2) + \overline{q}_i(x, Q^2) \right)$$

nonsinglet

$$q_i^{NS}(x, Q^2) = q_i(x, Q^2) - \overline{q}_i(x, Q^2)$$

DGLAP Evolution Equations

$$Q^2 \frac{d}{dQ^2} q^{NS}(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} P_{qq} \otimes q^{NS}(Q^2)$$

$$Q^2 \frac{d}{dQ^2} q^{S}(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{qq} \otimes q^{S}(Q^2) + 2n_f P_{qG} \otimes G(Q^2) \right]$$

$$Q^2 \frac{d}{dQ^2} G(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{Gq} \otimes q^{S}(Q^2) + P_{GG} \otimes G(Q^2) \right]$$

DGLAP for Mellin moments

Moments of the convolution

$$M_n = \int_0^1 dx\, x^{n-1} P \otimes f = \int_0^1 dx\, x^{n-1} \int_0^1 dz \int_0^1 dy \delta(zy - x) P(z) f(y)$$

$$= \int_0^1 dz\, z^{n-1} P(z) \int_0^1 dy\, y^{n-1} f(y) = P_n\, f_n = \gamma^n f_n$$

γ^n anomalous dimension

convolution is replaced by a product

DGLAP for Mellin moments

$$\frac{dq_n^{NS}(t)}{dt} = \frac{\alpha_s(t)}{2\pi}\gamma_{qq}^n q_n^{NS}(t)$$

$$\frac{d}{dt}\left[\begin{array}{c} q_n^S(t) \\ \\ G_n(t) \end{array}\right] = \frac{\alpha_s(t)}{2\pi}\left[\begin{array}{cc} \gamma_{qq}^n & 2n_f\gamma_{qG}^n \\ \gamma_{Gq}^n & \gamma_{GG}^n \end{array}\right]\left[\begin{array}{c} q_n^S(t) \\ \\ G_n(t) \end{array}\right]$$

$$\frac{\alpha_s(t)}{2\pi} = 2\,a_s(t) = 2\,\frac{1}{\beta_0 t}$$

Anomalous dimensions

$$\gamma_{qq}^{n} = C_F \left[-2 \sum_{k=1}^{n+1} \frac{1}{k} + \frac{3}{2} + \frac{1}{n} + \frac{1}{n+1} \right],$$

$$\gamma_{qG}^{n} = \frac{1}{2} \frac{2+n+n^2}{n(n+1)(n+2)},$$

$$\gamma_{Gq}^{n} = C_F \frac{2+n+n^2}{n(n^2-1)}$$

$$\gamma_{GG}^{n} = 2C_A \left[\frac{11}{12} - \sum_{k=1}^{n+2} \frac{1}{k} + \frac{1}{n-1} - \frac{1}{n} + \frac{2}{n+1} \right] - \frac{n_f}{3}$$

Valnce quark # conservation

$$\gamma_{qq}^{n} = C_F \left[-2\sum_{k=1}^{n+1}\frac{1}{k} + \frac{3}{2} + \frac{1}{n} + \frac{1}{n+1} \right]$$

$$\gamma_{qq}^{1} = 0 \quad \rightarrow \quad \frac{dq_n^{NS}(t)}{dt} = 0$$

$$\int dx \left[q_i(x,Q^2) - \overline{q}_i(x,Q^2) \right] = \text{const.} = \int dx q_{Vi}(x,Q^2)$$

Anomalous dimensions $n = 2$

$$\gamma_{qq}^{n} = C_F \left[-2 \sum_{k=1}^{n+1} \frac{1}{k} + \frac{3}{2} + \frac{1}{n} + \frac{1}{n+1} \right] \rightarrow -\frac{4C_F}{3}$$

$$\gamma_{qG}^{n} = \frac{1}{2} \frac{2 + n + n^2}{n(n+1)(n+2)} \rightarrow \frac{1}{6}$$

$$\gamma_{Gq}^{n} = C_F \frac{2 + n + n^2}{n(n^2 - 1)} \rightarrow \frac{4C_F}{3}$$

$$\gamma_{GG}^{n} = 2C_A \left[\frac{11}{12} - \sum_{k=1}^{n+2} \frac{1}{k} + \frac{1}{n-1} - \frac{1}{n} + \frac{2}{n+1} \right] - \frac{n_f}{3}$$

$$\rightarrow 0$$

Momentum conservation

consider moment $n = 2$ for the singlet eqs.

$$\frac{d}{dt} q_2^S(t) = -\frac{2}{\beta_0 t}\left[\frac{4C_F}{3} q_2^S(t) - \frac{n_f}{3} G_2(t)\right] = -\frac{2}{\beta_0 t} f(t)$$

$$\frac{d}{dt} G_2(t) = +\frac{2}{\beta_0 t}\left[\frac{4C_F}{3} q_2^S(t) - \frac{n_f}{3} G_2(t)\right] = +\frac{2}{\beta_0 t} f(t)$$

$$q_2^S(t) + G_2(t) = \text{const.}$$

$$= \int dx\, x \left[\sum_i \left(q_i(x, Q^2) + \overline{q}_i(x, Q^2)\right) + G(x, Q^2)\right] = 1$$

value of 1 is a requirement for a proper normalization

Gluon momentum

$$\frac{d}{dt}q_2^S(t) = -\frac{2}{\beta_0 t}\left[\frac{4C_F}{3}q_2^S(t) - \frac{n_f}{3}G_2(t)\right] = -\frac{2}{\beta_0 t}f(t)$$

$$\frac{d}{dt}G_2(t) = +\frac{2}{\beta_0 t}\left[\frac{4C_F}{3}q_2^S(t) - \frac{n_f}{3}G_2(t)\right] = +\frac{2}{\beta_0 t}f(t)$$

Form a linear combination

$$\frac{4C_F}{3}\frac{d}{dt}q_2^S(t) - \frac{n_f}{3}\frac{d}{dt}G_2(t) = \frac{d}{dt}f(t) = -\frac{2}{\beta_0 t}\left[\frac{4C_F}{3} + \frac{n_f}{3}\right]f(t)$$

since $c = \frac{4C_F}{3} + \frac{n_f}{3} > 0$

the solution is trivial and tends to 0 $f(t) = f(t_0)\left(\frac{t}{t_0}\right)^{-2c/\beta_0} \underset{t\to\infty}{\longrightarrow} 0$

Gluon momentum

We have two asymptotic constraints:

$$f(t) = \frac{4C_F}{3} q_2^S(t) - \frac{n_f}{3} G_2(t) = 0 \qquad\qquad q_2^S(t) + G_2(t) = 1$$

which give

$$q_2^S(t) = \frac{n_f}{4C_F} G_2(t) \qquad \to \qquad \left[\frac{n_f}{4C_F} + 1\right] G_2(t) = 1$$

numerically we have

$$G_2(t) = \frac{1}{\frac{n_f}{4C_F} + 1} = \frac{16}{16 + 3n_f} = \underset{n_f=3}{0.64}, \underset{n_f=4}{0.57}, \underset{n_f=5}{0.52}, \underset{n_f=6}{0.47}$$

asymptotically gluons carry around 50% of total momentum!