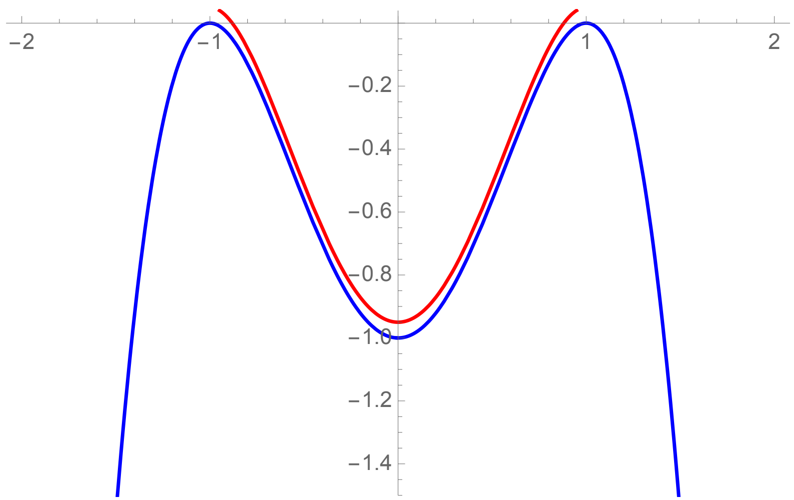
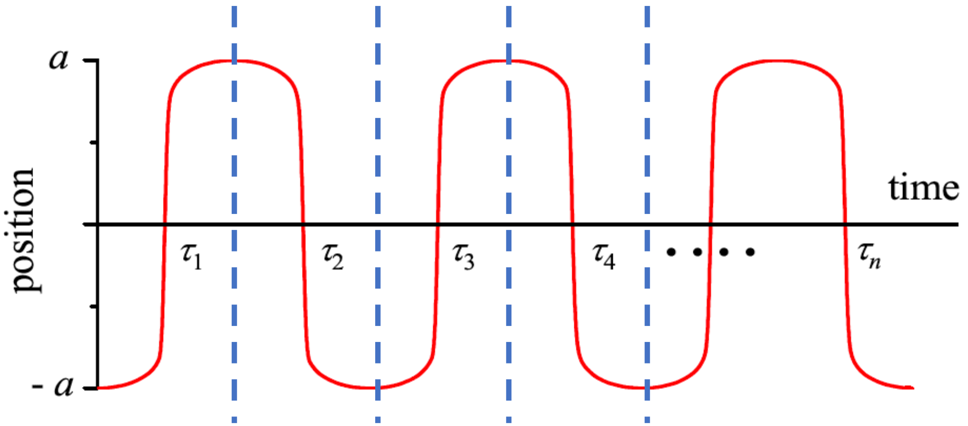
QCD lecture 11a

December 20

Energy splitting from instantons in QM





Two lowest energies

$$E_s = \frac{1}{2}\hbar\omega - \hbar\tilde{K}e^{-S_E^0/\hbar}$$

$$E_r = \frac{1}{2}\hbar\omega + \hbar\tilde{K}e^{-S_E^0/\hbar}$$

$$\tilde{K} = \frac{[\det(-m\frac{d^2}{d\tau^2} + m\omega^2))]^{\frac{1}{2}}}{[\det(-m\frac{d^2}{d\tau^2} + V''(\bar{x}))]^{\frac{1}{2}}}$$

Splitting is nonperturbative suppressed by the exponent from the classical action

Instanton in QM: summary

$$\tilde{K} = \left(\frac{S_E^0}{m2\pi\hbar}\right)^{\frac{1}{2}} \frac{[\det(-m\frac{d^2}{d\tau^2}+m\omega^2))]^{\frac{1}{2}}}{[\det{}'(-m\frac{d^2}{d\tau^2}+V''(\bar{x}))]^{\frac{1}{2}}}$$

Here prime means: no zero mode

Instantons in Minkowski space correspond to the tunelling between the minima of the potential.

In Euclidean space instantons are *localized* (around τ_1) solutions of classical equations of motion that in infinty go to the different vacua.

Instanton quantal operator for fluctuations around classical trajectory has a zero mode.

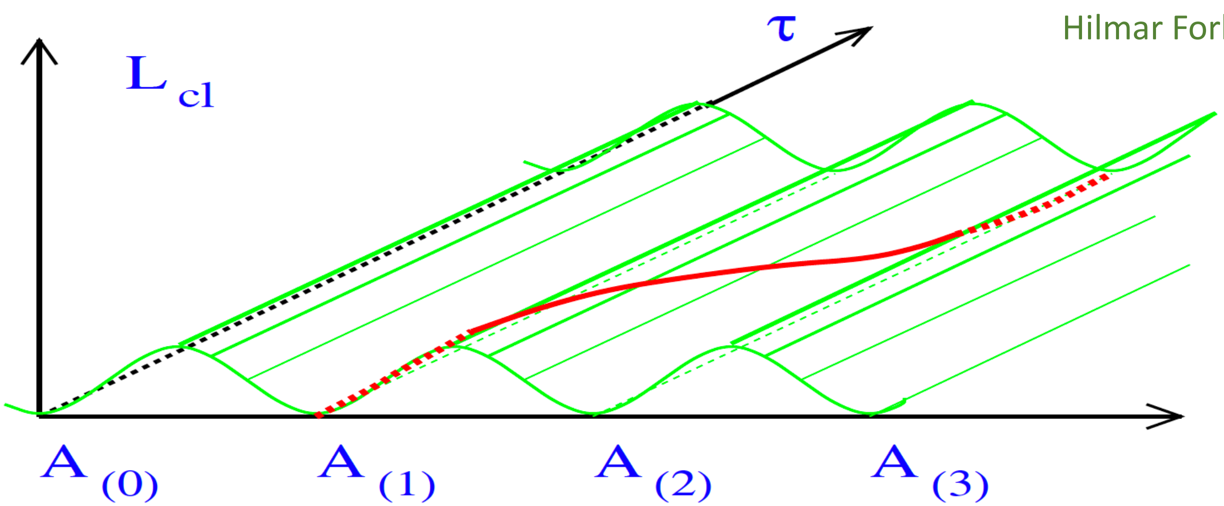
Zero modes have to be omitted from the quantal determinant and taken care off exactly.

Instantons give rise to the splitting of naively degenarate energy eigen-states. This splitting is non-perturbative and exponenially suppressed.

Instantons in QCD

In order to continously deform $A_\mu^{(n)} \to A_\mu^{(m)}$ we have to consider field configurations with nonminimal action $S_E > 0$

Hilmar Forkel *A Primer on Instnatons*



Instantons are solutions of the Euclidean equations of motion (QCD or Yang Mills eqs.)

$$D_\mu^{ab} F_{\mu\nu}^b = 0$$

with the following boundary conditions:

$$A_\mu(\vec{x}, T = -\infty) = A_\mu^{(n)}(\vec{x}),$$
$$A_\mu(\vec{x}, T = +\infty) = A_\mu^{(n+1)}(\vec{x})$$

They are time dep. solutions of $n = 1$ and minimal possible action.

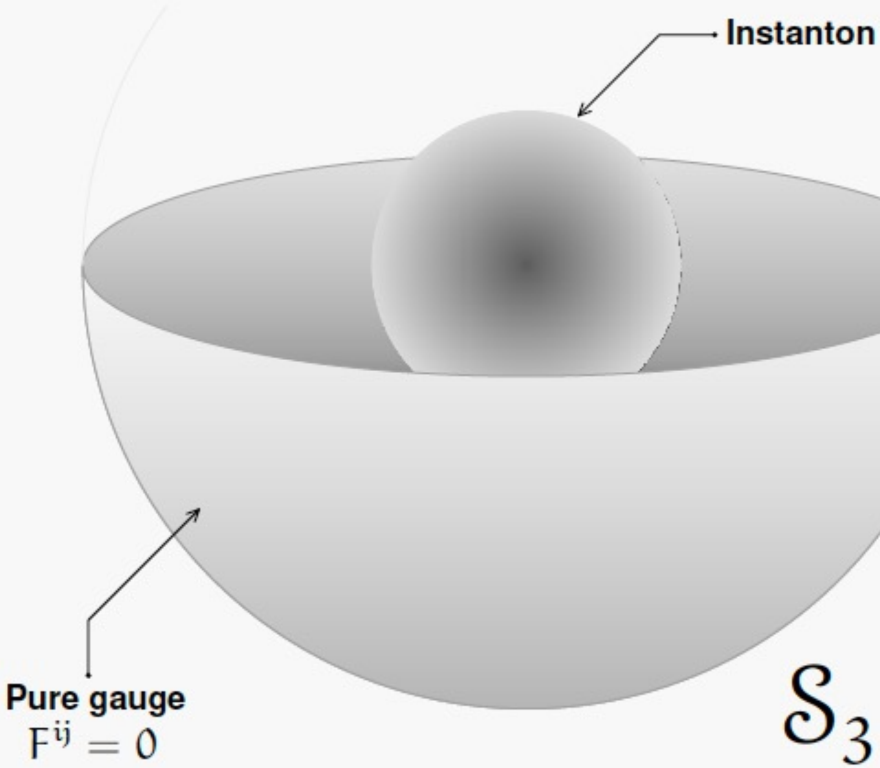


Figure 10.8: *Cartoon of an instanton (the illustration is for* $D = 3$*, although instantons actually exist in* $D = 4$*). The sphere* $\mathcal{S}_3$ *is in fact infinitely far away from the center of the instanton.*

F. Gelis, A Stroll Through Quantum Fields

Instantons in QCD

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Instantons in QCD

Instantons satisfy important property. Define dual field tensor $\tilde{F}^a_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^a_{\alpha\beta}$

Recall: $$\frac{g^2}{32\pi^2}\int d^4x\,\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta} = \frac{g^2}{32\pi^2}\int d^4x\,F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = N_W \;(=1)$$

$$S = \frac{1}{4}\int d^4x\,F^a_{\mu\nu}F^a_{\mu\nu}$$ Euclidean action has +sign

Construct a positive quantity:

$$0 \le \int d^4x\,\left(F^a_{\mu\nu} \pm \tilde{F}^a_{\mu\nu}\right)^2 = \int d^4x\,\left(2F^a_{\mu\nu}F^a_{\mu\nu} \pm 2F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}\right) = 8S \pm \frac{64\pi^2}{g^2}N_W$$

which gives a Bogomolny bound

$$S \ge \frac{8\pi^2}{g^2}\,|N_W|$$

Instantons minimize the action, so they are self-dual solutions $F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu}$

Explicit instanton solution SU(2)

$$A^a_\mu(x) = \frac{2}{g}\eta^a_{\mu\nu}\frac{(x-z)_\nu}{(x-z)^2+\rho^2}$$

't Hooft symbols

$$\eta^a_{\mu\nu} = \begin{cases} \varepsilon^{a\mu\nu} & \mu,\nu = 1,2,3 \\ -\delta^{a\nu} & \mu = 4 \\ +\delta^{a\mu} & \nu = 4 \\ 0 & \mu=\nu=4 \end{cases}$$

change sign for anti-instantons

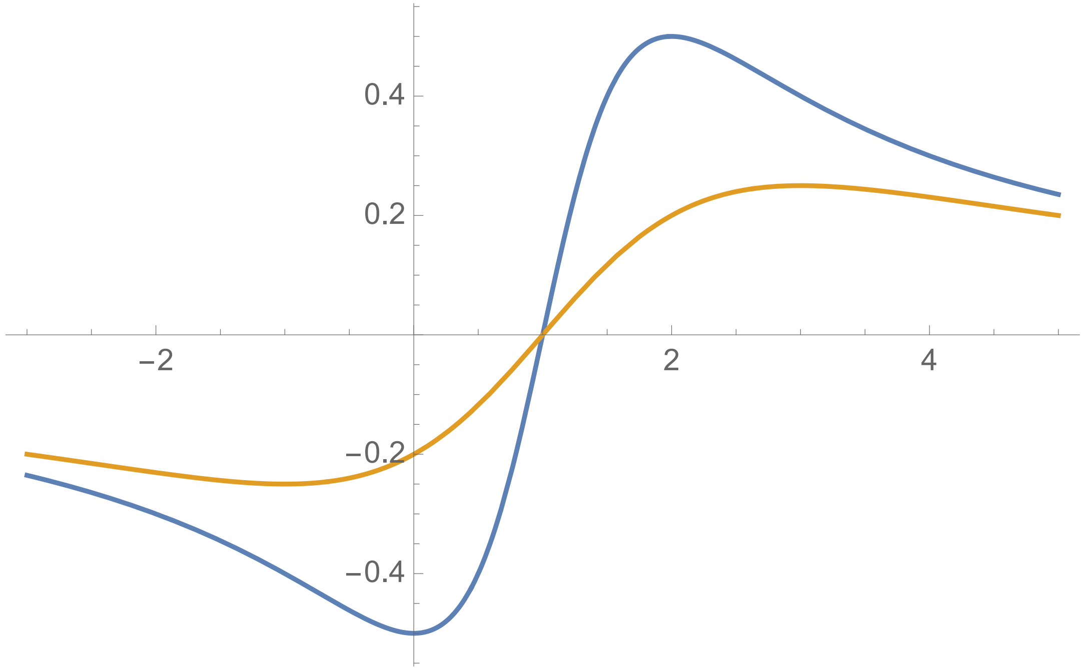
$$\eta^1_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \eta^2_{\mu\nu} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \eta^3_{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

They are localized solutions. z_μ is called instanton center, ρ instanton size

Explicit instanton solution

$$A^a_\mu(x) = \frac{2}{g}\eta^a_{\mu\nu}\frac{(x-z)_\nu}{(x-z)^2+\rho^2}$$

One dim. plot:



They are localized solutions. z_μ is called instanton center, ρ instanton size

Collective coordinates

Once we have classical solution we have to calculate quantal deteminant. However, like in QM, there will be zero modes corresponding to the flat directions of the classical action (within a topological class):

- change of instnanton center z_μ - 4
- change of size ρ - 1
- 3 parameters of a global gauge transformation (or 3 rotations)

Therefore there are 8 zero modes. In QM we had one zero mode corresponding to τ_1

Consider fluctuations around the classical configuration

$$A = A^{\rm inst} + a$$

Then

$$S[A^{\rm inst} + a] \;=\; \frac{8\pi^2}{g^2} + \frac{1}{2}\int d^4x \int d^4y\, a(x)\mathcal{D}(x,y)a(y)$$

$$\rightarrow\; e^{-8\pi^2/g^2}\frac{1}{\sqrt{\det \mathcal{D}}} = e^{-8\pi^2/g^2}\prod_s \lambda_s^{-1/2}$$

Collective coordinates

$$e^{-8\pi^2/g^2}\frac{1}{\sqrt{\det \mathcal{D}}} = e^{-8\pi^2/g^2}\prod_s \lambda_s^{-1/2}$$

However, for each zero mode we do not integrate over a complete set of eigen-functions of $\mathcal{D}$ but w perform an exact integration over the zero modes. Recall that there is a Jacobian between the two. In QM we had

$$\sqrt{\frac{S_E^0}{m}}\, y_1\, d\tau_1 \quad \text{instead of } y_1 da_1$$

This holds also in the QCD case. It can be shown by field rescaling that λ ~ $1/g^2$

$$S = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu}$$

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + g f^{abc} A^b_\mu A^c_\nu \\ &= \frac{1}{g}\left(\partial_\mu(gA_\nu) - \partial_\nu(gA_\mu) + f^{abc}(gA^b_\mu)(gA^c_\nu)\right) \\ &= \frac{1}{g}\left(\partial_\mu A'_\nu - \partial_\nu A'_\mu + f^{abc} A'^b_\mu A'^c_\nu\right). \end{aligned}$$

$$A' = gA$$

Collective coordinates

$$e^{-8\pi^2/g^2}\frac{1}{\sqrt{\det\mathcal{D}}} = e^{-8\pi^2/g^2}\prod_s \lambda_s^{-1/2}$$

However, for each zero mode we do not integrate over a complete set of eigen-functions of $\mathcal{D}$ but w perform an exact integration over the zero modes. Recall that there is a Jacobian between the two. In QM we had

$$\sqrt{\frac{S_E^0}{m}}\, y_1\, d\tau_1 \quad \text{instead of } y_1 da_1$$

This holds also in the QCD case. It can be shown by field rescaling that $\lambda \sim 1/g^2$

$$e^{-8\pi^2/g^2}\prod_{\substack{\text{zero}\\ \text{modes}}}\frac{1}{g}\prod_{\substack{s\neq\text{zero}\\ \text{mode}}}\lambda_s^{-1/2} \sim e^{-8\pi^2/g^2}\frac{1}{g^8}$$

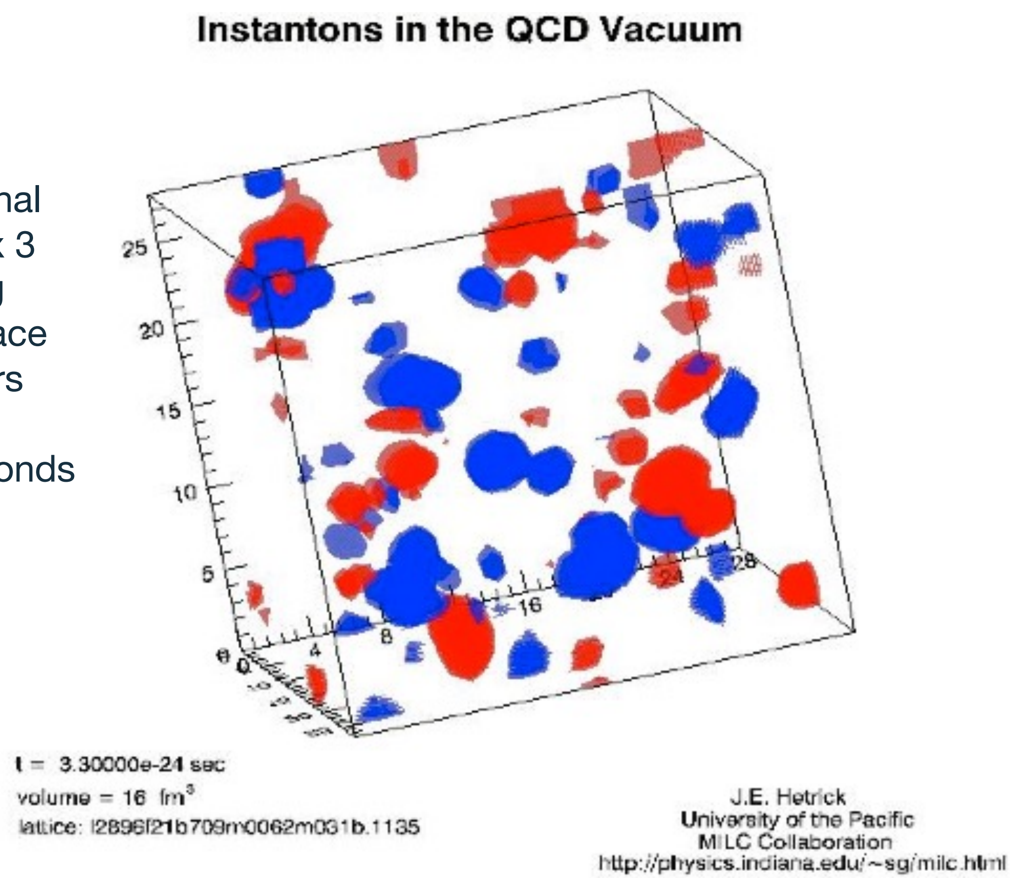
This shows how highly non-perturbative are instanton contributions to the expectation values in QCD.

Nonzero modes make coupling constant running.

Lattice QCD

Each lattice is a four-dimensional array (283 x 96, say) of four 3 x 3 complex matrices representing these fields in a tiny box of space measuring about 2 femtometers on a side (1 fm = 10^{-15} m) and extending about 10^{-22} seconds in time.

Instantons and anti-instantons

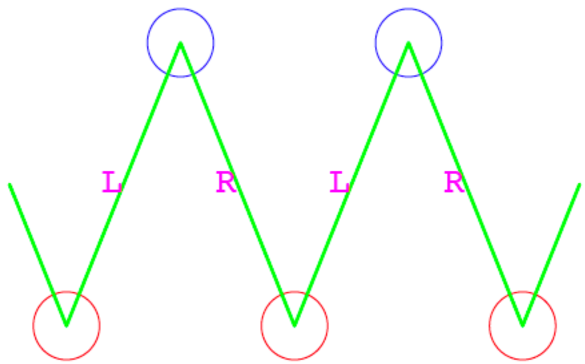


https://www.nersc.gov/news-publications/nersc-news/nersc-centernews/2004/nersc-s-qcd-library-is-a-model-resource/

Chiral symmetry breaking

$$\frac{g^2}{32\pi^2}\int d^4x_E\ \epsilon_{ijkl}\, F^a_{ij}(x)\, F^b_{kl}(x)\, \mathrm{tr}(t^a t^b) = n_R - n_L$$

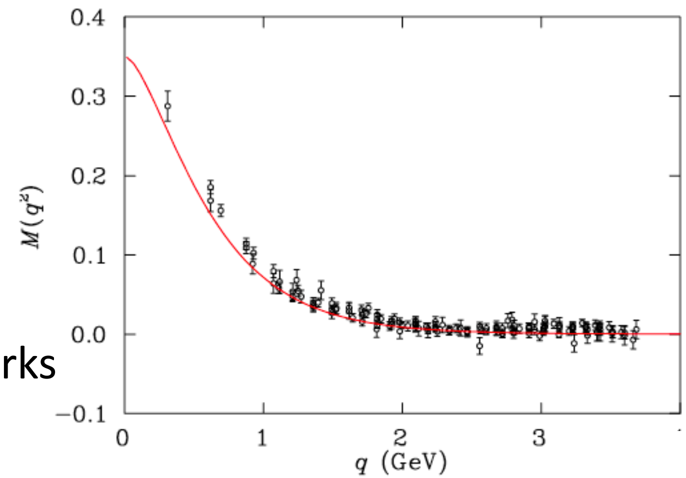
Quark propagaring between instantons and anti-instantons changes chirality.



Average instanton zize ρ = 1/3 fm and R = 1 fm (average distance between instantons)

Diakonov 2003
Instantons at work

This leads to the chiral symmetry breaking, quarks get constituent mass that is momentum dependent



This mechanism explains why proton has mass of 1 GeV, while current (Higgs) masses of u,d quarks are ~a few MeV

$$\epsilon_{vac} \simeq -\frac{b_1}{128\pi^2}\langle 0\left|g^2G^2\right|0\rangle \simeq -\frac{1}{2}\frac{\text{GeV}}{\text{fm}^3}$$

Instantons and θ term

In principle we should include the sum over all topological sectors in the QCD path integral

$$\langle \mathcal{O} \rangle = Z^{-1} \sum_{n \in \mathbb{Z}} P(n) \int [DA]_n \, \mathcal{O}[A] \, e^{-\mathcal{S}[A]}$$

where $P(n)$ is a weight factor and measure $[DA]_n$ is restricted to topological sector n. One can prove

$$P(n_1 + n_2) = P(n_1) P(n_2)$$

the solution is $P(n) = e^{-n\theta}$ where θ is an arbitrary constant. However

$$n = \frac{g^2}{64\pi^2} \int d^4x \, \epsilon^{ijkl} F^a_{ij} F^a_{kl}$$

So we may add theta term to the QCD lagrangian and integrate over all A fields. Note that $\theta = 0$ corresponds to the uniform weight factor.

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