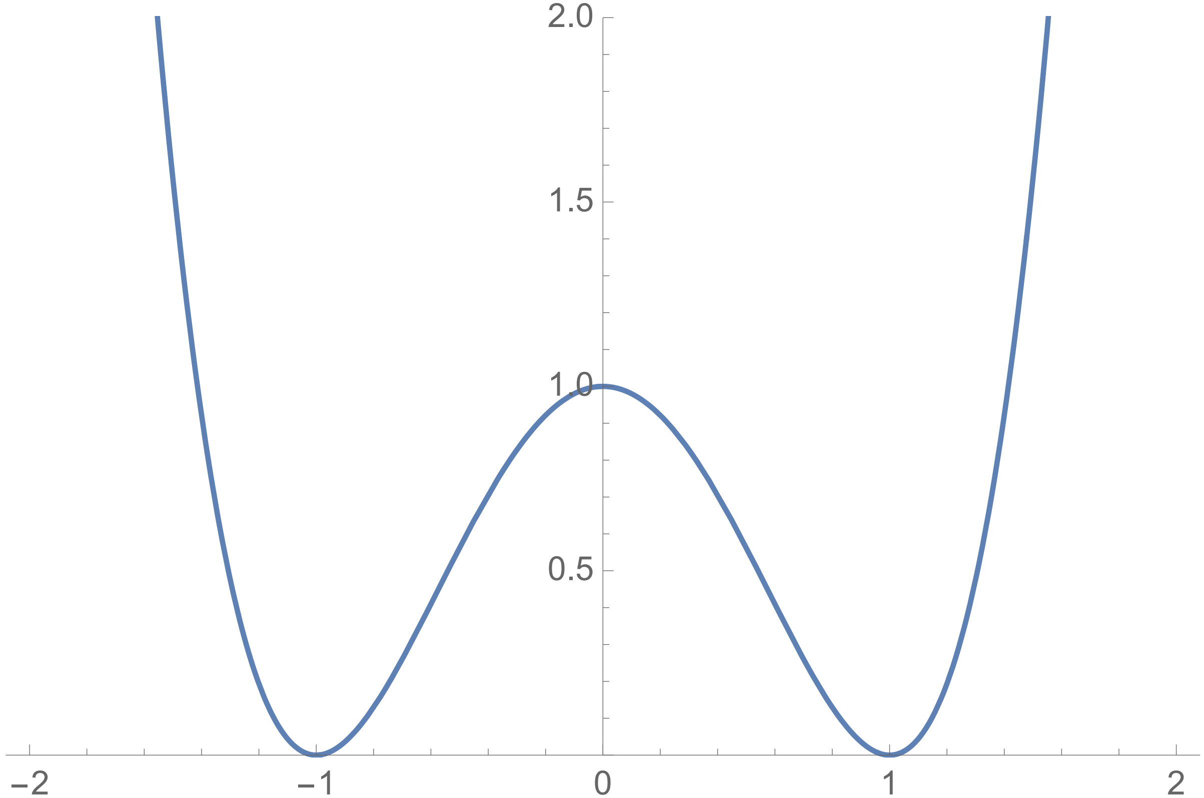
QCD lecture 10

December 13

Double well potential



Euclidean path integral

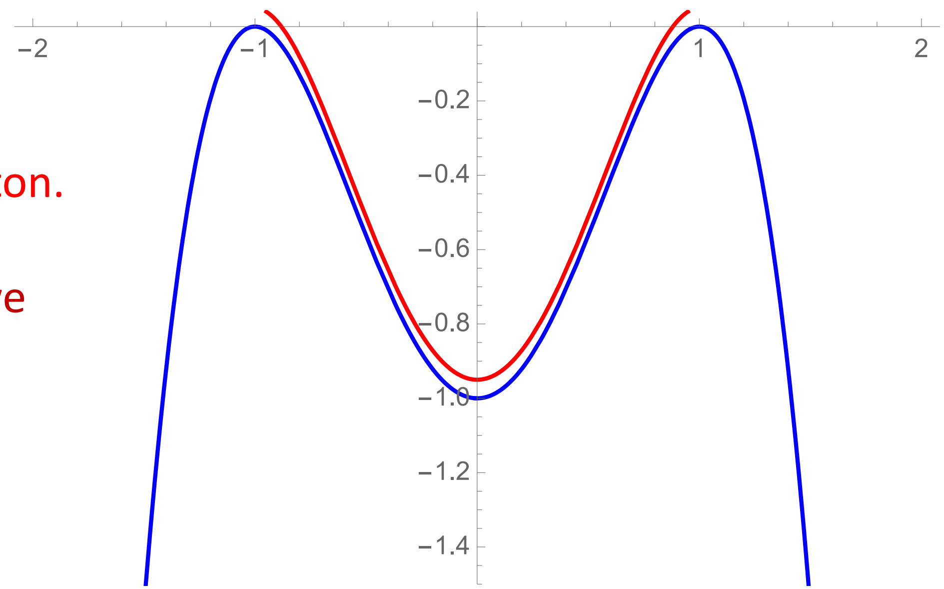
There is no classical trajectory: $-a \to a$ Go to Euclidean time $t = -i\tau$
where

$$K_E(x_b, \tfrac{1}{2}T; x_a, -\tfrac{1}{2}T) = <x_a|e^{-\frac{1}{\hbar}HT}|x_a> = \int [\mathcal{D}_E x(\tau)] e^{-\frac{1}{\hbar}S_E[x(\tau)]}$$

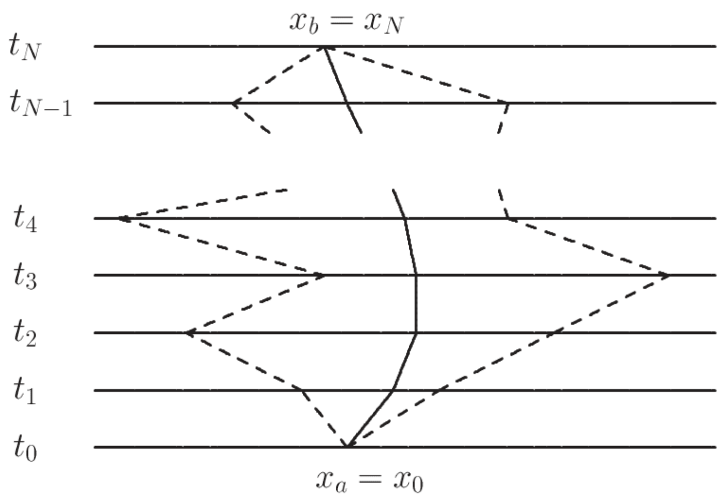
$$S_E[x(\tau)] = \int_{-T/2}^{T/2} d\tau \left[\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]$$

Potential is inverted and
there is
a classical trajectory called instanton.

To calculate the energy splitting we
have to sum over an infinite
number of instantons



Path integral in QM – reminder



$$K(x_b, x_a, t_b - t_a) = F(t_b - t_a)\, e^{\frac{i}{\hbar} S[\bar{x}(t)]}$$

$$\delta^2 S \;=\; -\int_0^T y\left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y}\right) + \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}}\right) y - \frac{\partial^2 L}{\partial x^2} y\right] dt \;=\; \int_0^T y\, D(t) y\, dt\,.$$

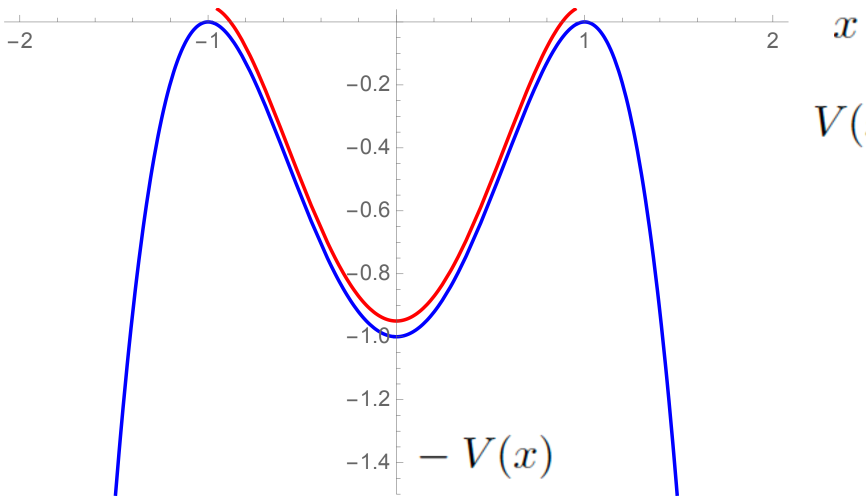
D is a Sturm-Liouville operator $\quad D(t) y_n(t) \;=\; \lambda_n y_n(t)\,, \quad n = 1, 2, 3, \ldots\,, \quad \lambda_1 < \lambda_2 < \ldots$

Use y_n basis to expand $\quad y(t) = \sum_{n=1}^{\infty} a_n\, y_n(t) \quad$ then $\quad \delta^2 S[y] = \sum_{n=1}^{\infty} \lambda_n a_n^2$

and $\quad [\mathcal{D}y(t)] \sim \prod_{n=1}^{\infty} da_n$

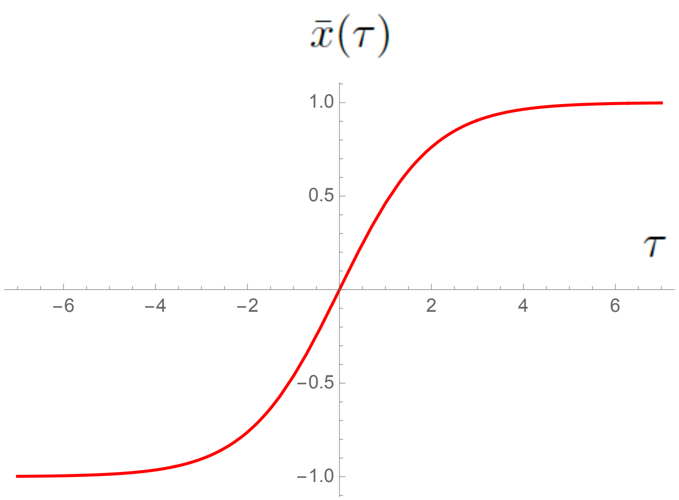
$$F(T) \sim \prod_{n=1}^{\infty} da_n \, \exp\left(\frac{i}{2\hbar} \lambda_n a_n^2\right) \sim \sqrt{\frac{1}{\prod_n \lambda_n}} = \sqrt{\frac{1}{\det D(t)}}$$

Explicit model



$$V(x) = \frac{1}{8a^2}(a^2 - x^2)^2$$

Instanton is an Euclidean trajectory of zero energy.



$$|\bar{x} - a| \sim e^{-\sqrt{\frac{V''(a)}{m}}\tau} = e^{-\omega\tau}$$

$$\bar{x}(\tau) = a \tanh\frac{\tau - \tau_1}{2}$$

Instanton – classical action

Recall that instanton has $E = 0$

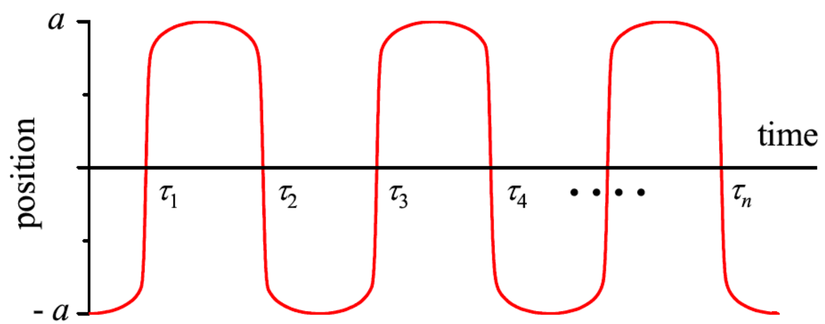
$$\tfrac{1}{2}m\dot{\bar{x}}^2 - V(x) = 0\,, \qquad \dot{\bar{x}} = \left[\frac{2}{m}V(\bar{x})\right]^{\frac{1}{2}}, \qquad \frac{d\tau}{d\bar{x}} = \frac{1}{\sqrt{\frac{2}{m}V(\bar{x})}}$$

Hence

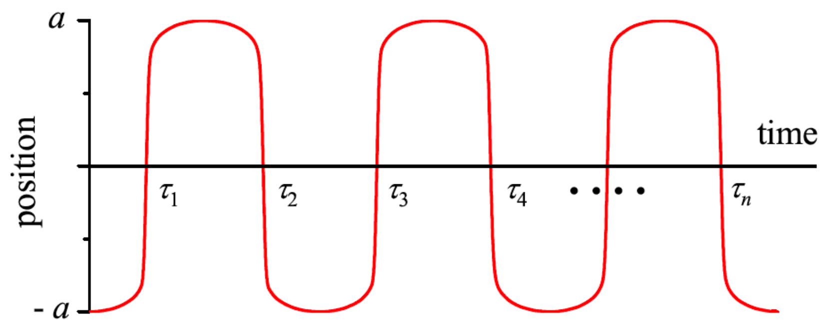
$$S_E^0 = \int_{-T/2}^{+T/2} d\tau \left[\tfrac{1}{2}m\frac{2}{m}V(\bar{x}) + V(\bar{x})\right] = \int_{-a}^{+a} d\bar{x}\sqrt{2mV(\bar{x})} = \int_{-a}^{+a} d\bar{x}\, p(\bar{x})$$

barrier transmission coefficient

Consider now amplitude $<\!-a|e^{-HT/\hbar}|-a\!>$ that has infnitely many jumps: instantons and anti-instantons separated in time (dilute approximation)



Multi-instanton transition amplitude



$$x(\tau) = \bar{x}_{\tau_1 \ldots \tau_n}(\tau) + y(\tau) \approx \bar{x}_{\tau_1}(\tau) + \bar{x}_{\tau_2}(\tau) + \ldots + \bar{x}_{\tau_n}(\tau) + y(\tau)$$

Here $\bar{x}_{\tau_1 \ldots \tau_n}(\tau)$ is the exact classical trajectory that can be approximated by a sum over one-(anti) instanton trajectories $\bar{x}_{\tau_n}$ where $\tau_1, \ldots, \tau_n$ mark times of individual jumps.

$$< -a|e^{-\frac{1}{\hbar}HT}| - a> = \sum_{\text{even } n} \int_{-T/2}^{T/2} d\tau_1 \ldots \int_{\tau_{n-2}}^{T/2} d\tau_{n-1} \int_{\tau_{n-2}}^{T/2} d\tau_n \, e^{-\frac{1}{\hbar}S_E[\bar{x}_{\tau_1 \ldots \tau_n}(\tau)]}$$

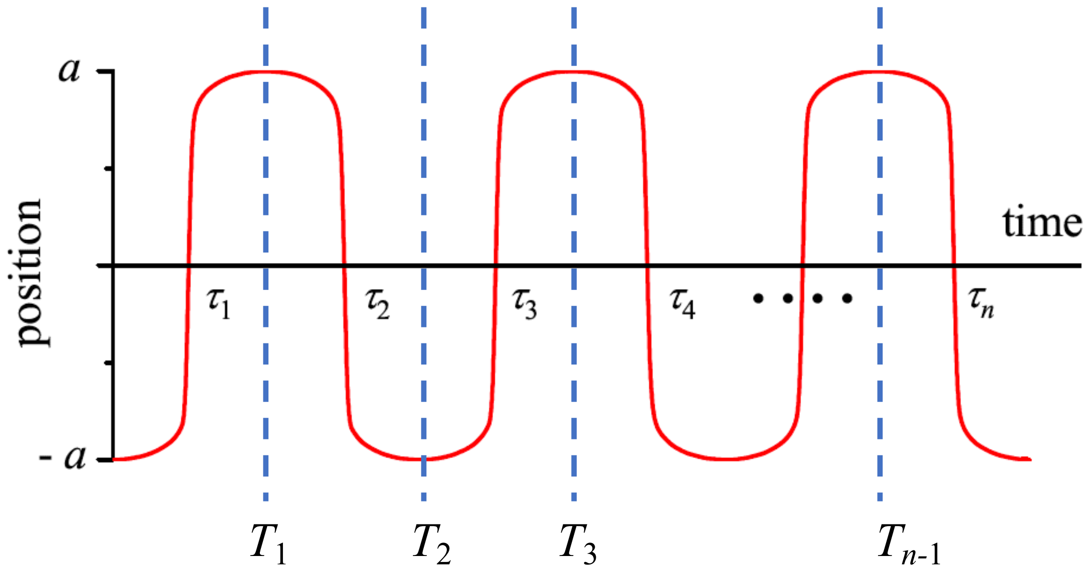
$$\times \int_{y(-\frac{T}{2})=y(+\frac{T}{2})=0} [\mathcal{D}y_E(\tau)] \, e^{-\frac{1}{2\hbar}\int_{-T/2}^{+T/2} d\tau y(-m\frac{d^2}{d\tau^2}+V''(\bar{x}))y}$$

Multi-instanton transition amplitude

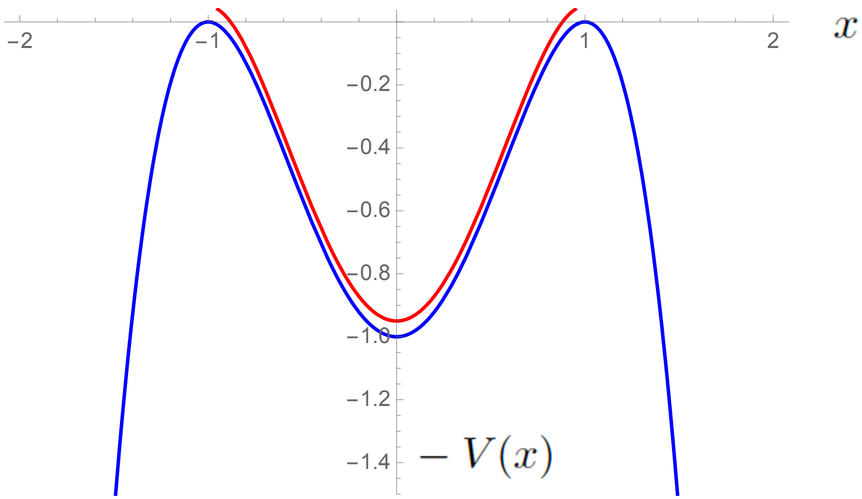
In dilute approximation $S_E[\bar{x}_{\tau_1 \ldots \tau_n}(\tau)] \approx n S_E^0$

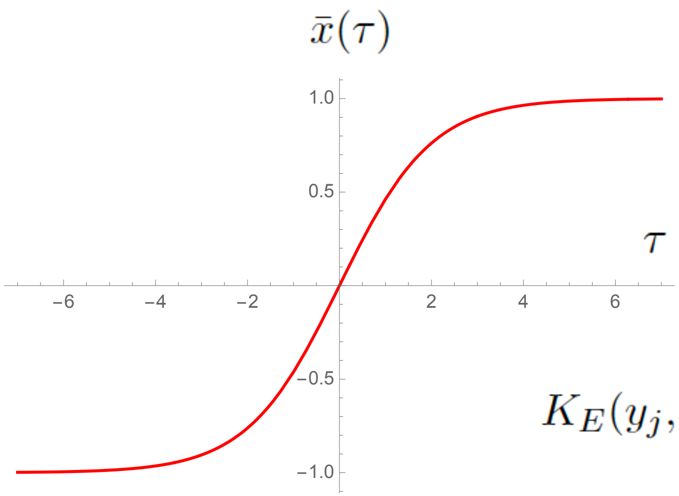
The quantal part can be written as a kind of propagator

$$K_E(0, \tfrac{1}{2}T; 0, -\tfrac{1}{2}T) = \int_{y(-\frac{T}{2})=y(+\frac{T}{2})=0} [\mathcal{D}y_E(\tau)]\, e^{-\frac{1}{2\hbar}\int_{-T/2}^{+T/2} d\tau\, y(-m\frac{d^2}{d\tau^2}+V''(\bar{x}))y}$$



Oscillator approximation





We are considering fluctuations around one instanton. But for most of the time the particle is either in one or the other maximum (minimum in Minkowski space) i.e. it sits there and does not move. This corresponds to a trivial classical trajectory of an Euclidean oscillator (potential is quadratic around each maximum). Quantal operator

$$\left(-m\frac{d^2}{d\tau^2}+V''(\bar{x})\right) \qquad \omega^2 = \frac{V''(\pm a)}{m}$$

is the same in either maximum. So we can approximate fluctuations around one instanton

$$K_E(y_j, T_j; y_{j-1}, T_{j-1}) = \tilde{K}\, K_E^{\rm osc}(y_j, T_j; y_{j-1}, T_{j-1})$$

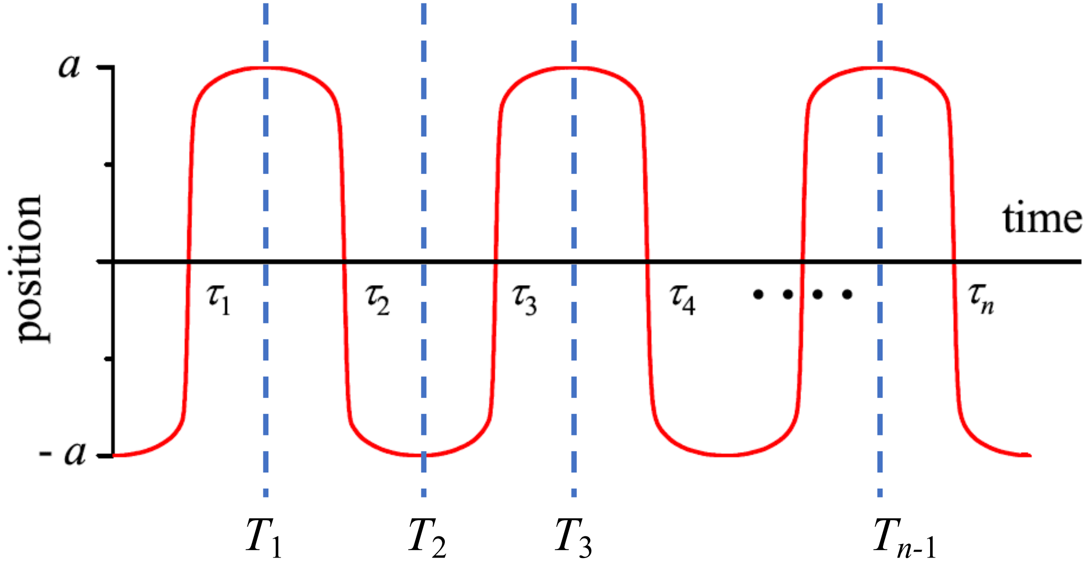
where $\tilde{K}$ is a correction factor.

Multi-instanton transition amplitude

In dilute approximation $S_E[\bar{x}_{\tau_1 \ldots \tau_n}(\tau)] \approx n S_E^0$

The quantal part can be written as a kind of propagator

$$\tilde{K}^n \int K_E^{\rm osc}(0, \frac{1}{2}T; y_{n-1}, T_{n-1})\, dy_{n-1} \ \ldots \ dy_2 K_E^{\rm osc}(y_2, T_2; y_1, T_1)\, dy_1 K_E^{\rm osc}(y_1, T_1; 0, -\frac{1}{2}T)$$

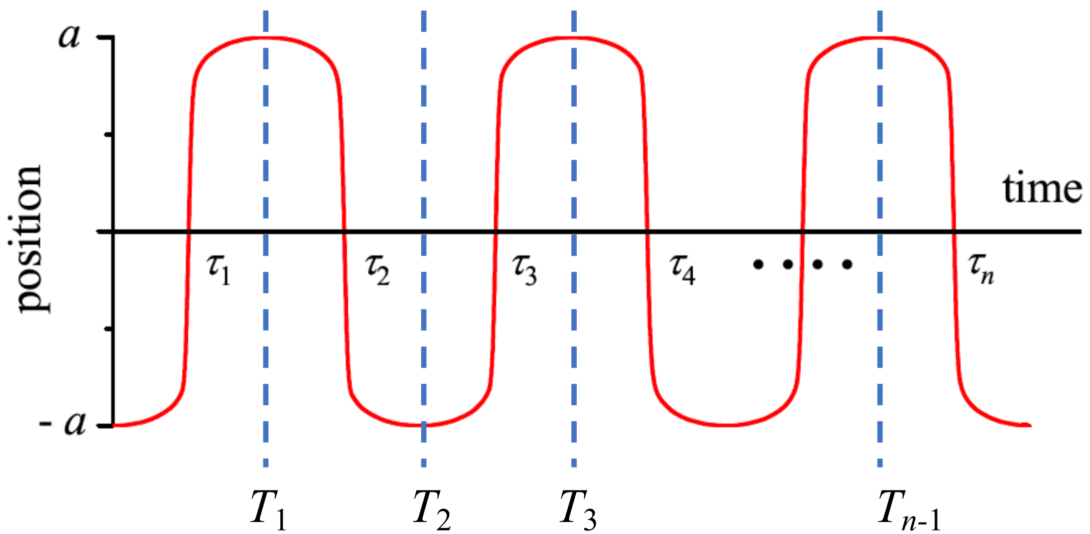


Multi-instanton transition amplitude

In dilute approximation $S_E[\bar{x}_{\tau_1\ldots\tau_n}(\tau)] \approx nS_E^0$

The quantal part can be written as a kind of propagator

$$\tilde{K}^n\, K_E^{\mathrm{osc}}(0, \frac{1}{2}T; 0, -\frac{1}{2}T)$$



Oscillator approximation

Recall energy representation for K

$$K(x_b, x_a, -i\tau) = <x_b|e^{-\frac{H}{\hbar}\tau}|x_a>$$

$$= \sum_{n,n'} <x_b|E_n><E_n|\, e^{-\frac{H}{\hbar}\tau}|E_{n'}><E_{n'}|x_a>$$

$$= \sum_n e^{-\frac{E_n}{\hbar}\tau}\phi_n(x_b)\phi_n^*(x_a)\,.$$

For large τ only the lowest level contributes, so we have

$$K_E(0, \tfrac{1}{2}T, 0, -\tfrac{1}{2}T)\Big|_{T\to\infty} = \tilde{K}^n \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\omega T}\,, \qquad \omega^2 = \frac{V''(\pm a)}{m}$$

Oscillator approximation

$$< -a|e^{-\frac{1}{\hbar}HT}| - a > = \sum_{\text{even}\, n} \int\limits_{-T/2}^{T/2} d\tau_1 \ldots \int\limits_{\tau_{n-2}}^{T/2} d\tau_{n-1} \int\limits_{\tau_{n-2}}^{T/2} d\tau_n \, e^{-\frac{1}{\hbar} S_E[\bar{x}_{\tau_1 \ldots \tau_n}(\tau)]}$$

$$\times \int_{y(-\frac{T}{2})=y(+\frac{T}{2})=0} [\mathcal{D} y_E(\tau)] \, e^{-\frac{1}{2\hbar} \int\limits_{-T/2}^{+T/2} d\tau y(-m \frac{d^2}{d\tau^2} + V''(\bar{x}))y}$$

Oscillator approximation

We started from

$$<-a|e^{-\frac{1}{\hbar}HT}|-a>=\sum_{\text{even}\,n}\int\limits_{-T/2}^{T/2}d\tau_1\ldots\int\limits_{\tau_{n-2}}^{T/2}d\tau_{n-1}\int\limits_{\tau_{n-2}}^{T/2}d\tau_n\,e^{-\frac{1}{\hbar}S_E[\bar{x}_{\tau_1\ldots\tau_n}(\tau)]}$$

$$\times\int_{y(-\frac{T}{2})=y(+\frac{T}{2})=0}[\mathcal{D}y_E(\tau)]\,e^{-\frac{1}{2\hbar}\int\limits_{-T/2}^{+T/2}d\tau y(-m\frac{d^2}{d\tau^2}+V''(\bar{x}))y}$$

Now we have

$$<-a|e^{-\frac{1}{\hbar}HT}|-a>=\sum_{\text{even}\,n}\int\limits_{-T/2}^{T/2}d\tau_1\ldots\int\limits_{\tau_{n-2}}^{T/2}d\tau_{n-1}\int\limits_{\tau_{n-1}}^{T/2}d\tau_n\,e^{-\frac{1}{\hbar}n\,S_E^0}\,\tilde{K}^n\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}e^{-\frac{1}{2}\omega T}$$

Since nothing depends on τ_i we can perform the integral (exercise)

$$\int\limits_{-T/2}^{T/2}d\tau_1\ldots\int\limits_{\tau_{n-2}}^{T/2}d\tau_{n-1}\int\limits_{\tau_{n-1}}^{T/2}d\tau_n\;=\;\frac{1}{n!}T^n$$

Energy splitting

$$<-a|e^{-\frac{1}{\hbar}HT}|-a> \approx \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\omega T} \sum_{\mathrm{even}\, n} \frac{1}{n!}\left(\tilde{K}e^{-S_E^0/\hbar}T\right)^n$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\omega T}\frac{1}{2}\left[e^{\tilde{K}e^{-S_E^0/\hbar}T} + e^{-\tilde{K}e^{-S_E^0/\hbar}T}\right]$$

$$= \frac{1}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \left[e^{-(\frac{1}{2}\omega-\tilde{K}e^{-S_E^0/\hbar})T} + e^{-(\frac{1}{2}\omega+\tilde{K}e^{-S_E^0/\hbar})T}\right]$$

Because in this limit only the the ground state survives, we have two lowest energies

$$E_s = \frac{1}{2}\hbar\omega - \hbar\tilde{K}e^{-S_E^0/\hbar}$$

$$E_r = \frac{1}{2}\hbar\omega + \hbar\tilde{K}e^{-S_E^0/\hbar}$$

Splitting is nonperturbative suppressed by the exponent from the classical action

Calculation of $\tilde{K}$

$$K_E(y_j, T_j; y_{j-1}, T_{j-1}) = \tilde{K}\, K_E^{\rm osc}(y_j, T_j; y_{j-1}, T_{j-1})$$

Note that $\tilde{K}$ is a number given by a ratio of the square root of two determinants

$$\tilde{K} = \frac{\left[\det\left(-m\frac{d^2}{d\tau^2} + m\omega^2\right)\right]^{\frac{1}{2}}}{\left[\det\left(-m\frac{d^2}{d\tau^2} + V''(\bar{x})\right)\right]^{\frac{1}{2}}}$$

For the instanton we need to find eigenvalues of

$$\left(-m\frac{d^2}{d\tau^2} + \frac{d^2V}{d\bar{x}^2}\right)y_n(\tau) = \lambda_n y_n(\tau)$$

We will show now that this operator has one zero mode (!). This would in principle render $\tilde{K}$ infinite.

Instanton zero mode

Recall that instanton has zero (Euclidean) energy:

$$\frac{1}{2}m\dot{\bar{x}}^2 - V(x) = 0 \quad \Rightarrow \quad \frac{d\bar{x}}{d\tau} = \left(\frac{2}{m}V(\bar{x})\right)^{\frac{1}{2}}$$

Let's differenciate velocity over time: $\frac{d^2\bar{x}}{d\tau^2} = \frac{d}{d\bar{x}}\left[\left(\frac{2}{m}V(\bar{x})\right)^{\frac{1}{2}}\right]\frac{d\bar{x}}{d\tau} = \frac{1}{m}\frac{dV}{d\bar{x}}$

and once more: $\left(-m\frac{d^2}{d\tau^2} + \frac{d^2V}{d\bar{x}^2}\right)\frac{d\bar{x}}{d\tau} = 0$

But this is our eigen-equation for a zero mode $\lambda_1 = 0$ (this the lowest eigen-value):

$$\left(-m\frac{d^2}{d\tau^2} + \frac{d^2V}{d\bar{x}^2}\right)y_n(\tau) = \lambda_n y_n(\tau)$$

We can normalize this mode (exercise)

$$y_1(\tau) = \left(S_E^0\right)^{-\frac{1}{2}}\sqrt{m}\,\frac{d\bar{x}}{d\tau} \qquad \text{where} \qquad S_E^0 = \int_{-T/2}^{+T/2} d\tau\, 2V(\bar{x})$$

Instanton zero mode

Consider one instnaton trajectory

$$x(\tau) = \bar{x}(\tau) + y(\tau) = \bar{x}(\tau) + a_1 y_1(\tau) + \sum_{l>1} a_l\, y_l(\tau) \qquad \int \prod_i da_i$$

Note that $\bar{x}(\tau) = \bar{x}_{\tau_1}(\tau) = \bar{x}(\tau - \tau_1)$

Change of the trajectory due to the change of the jump time τ_1 is equal to

$$dx(\tau) = \frac{d\bar{x}}{d\tau_1}\, d\tau_1 = -\frac{d\bar{x}}{d\tau}\, d\tau_1 = -\sqrt{\frac{S_E^0}{m}}\, y_1\, d\tau_1$$

But this is the change corresponding to the zero mode

$$dx(\tau) \;=\; y_1 da_1$$

So we have already taken this change into accound when integrating over jump times. This is the exact result (while integrations over da_i are in Gaussian approximation). We therefore have to omit $\lambda_1 = 0$ in the instanton determinant, include Jacobian for the change of variables and remove $\sqrt{2\pi\hbar}$ arising from the Gaussian integral.

Instanton in QM: summary

$$\tilde{K} = \left(\frac{S_E^0}{m2\pi\hbar}\right)^{\frac{1}{2}} \frac{[\det(-m\frac{d^2}{d\tau^2} + m\omega^2))]^{\frac{1}{2}}}{[\det{}'(-m\frac{d^2}{d\tau^2} + V''(\bar{x}))]^{\frac{1}{2}}}$$

Here prime means: no zero mode

Instantons in Minkowski space correspond to the tunelling between the minima of the potential.

In Euclidean space instantons are *localized* (around τ_1) solutions of classical equations of motion that in infinty go to the different vacua.

Instanton quantal operator for fluctuations around classical trajectory has a zero mode.

Zero modes have to be omitted from the quantal determinant and taken care off exactly.

Instantons give rise to the splitting of naively degenarate energy eigen-states. This splitting is non-perturbative and exponenially suppressed.