QCD
problem set 14

1. In the SU(3) case the mass term in the effective lagrangian is equal to (up to a constant term)
$$\mathcal{L}_{\rm eff}^{(m)} = -\text{const.}\,\text{Tr}(UM^\dagger + MU^\dagger)$$
where the quark mass matrix reads.
$$M = \text{diag}[m_u, m_d, m_s].$$
Decomposition of the mass matrix in terms of λ_0, λ_3 and λ_8 can be found in lecture 12. Matrix $U = \exp(i\phi/F)$ can be expressed in terms of the physical meson fields:
$$\phi(x) = \sum_a \lambda_a \phi^a(x) = \begin{pmatrix} \pi^0 + \eta/\sqrt{3} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \eta/\sqrt{3} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -2\eta/\sqrt{3} \end{pmatrix}.$$
Calculate the mass term (up to the quadratic terms in fields) and interpret all terms. Next, assume $m_u = m_d = m$ and calculate meson masses. There will be three masses for pions, kaons and eta expressed in terms of two parameters const.$\times m$ and const.$\times m_s$. Therefore there will be one, parameter independent, relation between these masses. Find this relation and check whether it is fulfilled experimentally.

2. There exists another possible parametrization of U in SU(2)
$$U = \frac{1}{F}\left[\sigma(x) + i\,\vec{\tau}\cdot\vec{\pi}(x)\right] \text{ where } \sigma(x) = \sqrt{F^2 - \vec{\pi}^2(x)}.$$
Calculate the effective lagrangian up to 4 fields in this case.

3. Canonical equal time commutation relation for scalar a field
$$\left[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{x}\,')\right] = i\delta^{(3)}(\vec{x} - \vec{x}\,')$$
where $\hat{\pi}(t,\vec{x}) = \partial_t\hat{\phi}(t,\vec{x})$ (why?) implies certain commutation rule for $\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')\right]$ where
$$\hat{\phi}(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left[e^{-i\,kx}\hat{a}(\vec{k}) + e^{+i\,kx}\hat{a}^\dagger(\vec{k})\right].$$
Find this commutation rule.

4. For on-shell heavy quark of mass M_Q one can write that its momentum is given as

$$p^\mu = M_Q v^\mu$$

provided $v^2 = 1$. Show that operators

$$\frac{1}{2}\left(1 \pm \not{v}\right)$$

satisfy all necessary conditions of projection operators.

5. Pove that

$$\frac{1+\not{v}}{2}\gamma^\mu \frac{1+\not{v}}{2} = \frac{1+\not{v}}{2} v^\mu \frac{1+\not{v}}{2}$$

6. Solutions of the Dirac equation in the Dirac (Bjorken-Drell) representation of γ matrices take the following form:

$$u(p,s) = \sqrt{E_p + m}\begin{bmatrix} \chi(s) \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m}\chi(s) \end{bmatrix}, \quad v(p,s) = \sqrt{E_p + m}\begin{bmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m}\chi(s) \\ \chi(s) \end{bmatrix}, \tag{1}$$

where $s = 1, 2$ labels spin. Solutions denoted by u correspond to particles, while solutions denoted by v to antiparticles (E_p is defined to be positive). Calculate action of the projection operators defined in the previous problem on these solutions for a heavy fermion at rest.