QCD
problem set 13

1. In the case of fermion fields, commutation relations of scalar fields are replaced by anticommutation relations:
$$\left\{q_{\alpha,k}(t,\vec{x}), q^\dagger_{\beta,l}(t,\vec{x}\,')\right\} = \delta^{(3)}(\vec{x}-\vec{x}\,')\delta_{\alpha\beta}\delta_{kl}$$
where α, β stand for Dirac indices and k, l denote SU(3) indices. Relevant charges are defined as
$$\begin{aligned}\hat{Q}^a_{L,R}(t) &= \int d^3\vec{x}\, q^\dagger_{L,R}(t,\vec{x})T^a q_{L,R}(t,\vec{x}),\\ \hat{Q}_V(t) &= \int d^3\vec{x}\,\left[q^\dagger_L(t,\vec{x})q_L(t,\vec{x}) + q^\dagger_R(t,\vec{x})q_R(t,\vec{x})\right]\end{aligned}$$
where $T^a = \lambda^a/a$ are SU(3) generators (Gell-Mann matrices). Making use of the identity (prove it!)
$$\{ab, cd\} = a\,\{b,c\}\,d - ac\{b,d\} + \{a,c\}\,bd - c\{a,d\}b$$
show that
$$\begin{aligned}\left[\hat{Q}^a_L, \hat{Q}^b_L\right] &= if^{abc}\hat{Q}^c_L,\\ \left[\hat{Q}^a_R, \hat{Q}^b_R\right] &= if^{abc}\hat{Q}^c_R,\\ \left[\hat{Q}^a_L, \hat{Q}^b_R\right] &= 0,\\ \left[\hat{Q}^a_{L,R}, \hat{Q}_V\right] &= 0.\end{aligned}$$

2. Effective Lagrangian describing Goldstone boson interactions in the chiral limit reads
$$\mathcal{L}_{\text{eff}} = \frac{F^2}{4}\,\text{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)$$
where $U = \exp(i\phi/F)$ can be expressed in terms of the meson fields ϕ^a:
$$\phi(x) = \sum_a \lambda_a \phi^a(x)$$
where $\lambda_a = \tau_a$ are Pauli matrices for SU(2) and Gell-Mann matrices for SU(3).

Expand $\mathcal{L}_{\text{eff}}$ up to 4-fields. You should get one term with two fields and two terms with four fields (still expressed in terms of flavor traces). Calculate flavor traces for the case of SU(2) and express them in terms of the physical pion fileds:
$$\phi(x) = \sum_a \lambda_a \phi^a(x) = \begin{bmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{bmatrix}$$

3. For the SU(2) case the mass term corresponding to the non-zero quark masses reads:

$$\mathcal{L}_{\rm eff}^{(m)} = -\frac{F^2 m_\pi^2}{4}\,{\rm Tr}(U + U^\dagger - 2)$$

Expand $\mathcal{L}_{\rm eff}^{(m)}$ up to 2-fields and calculate flavor trace.

4. In the SU(3) case the mass term in the effective lagrangian is equal to (up to a constant term)

$$\mathcal{L}_{\rm eff}^{(m)} = -{\rm const.}\,{\rm Tr}(UM^\dagger + MU^\dagger)$$

where the quark mass matrix reads.

$$M = {\rm diag}[m_u, m_d, m_s].$$

Decomposition of the mass matrix in terms of λ_0, λ_3 and λ_8 can be found in lecture 12. Matrix $U = \exp(i\phi/F)$ can be expressed in terms of the physical meson fields:

$$\phi(x) = \sum_a \lambda_a \phi^a(x) = \left(\begin{array}{ccc} \pi^0 + \eta/\sqrt{3} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \eta/\sqrt{3} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -2\eta/\sqrt{3} \end{array} \right).$$

Calculate the mass term (up to the quadratic terms in fields) and interpret all terms. Next, assume $m_u = m_d = m$ and calculate meson masses. There will be three masses for pions, kaons and eta expressed in terms of two parameters const.$\times m$ and const.$\times m_s$. Therefore there will be one, parameter independent, relation between these masses. Find this relation and check whether it is fulfilled experimentally.

5. There exists another possible parametrization of U in SU(2)

$$U = \frac{1}{F}\left[\sigma(x) + i\,\vec{\tau}\cdot\vec{\pi}(x)\right] \text{ where } \sigma(x) = \sqrt{F^2 - \vec{\pi}^{\,2}(x)}.$$

Calculate the effective lagrangian up to 4 fields in this case.