

QCD
problem set 10

1. Winding number of the SU(2) gauge transformation U is defined as

$$N_w = \frac{1}{24\pi^2} \varepsilon^{ijk} \int d^3r \operatorname{Tr} [(U^\dagger \partial_i U) (U^\dagger \partial_j U) (U^\dagger \partial_k U)]. \quad (1)$$

Calculate (1) for $U = \exp(i \vec{n} \cdot \vec{\tau} P(r))$ where $\vec{n} = \vec{r}/r$. What are the boundary conditions for $P(r)$ that ensure that N_w is an integer?

HINT:

First decompose

$$U^\dagger \partial_i U = \frac{i}{2} \sum_{a=1}^3 \xi_i^a \tau_a$$

where τ_a are Pauli matrices. You should obtain that $\varepsilon^{ijk} \operatorname{Tr} [(U^\dagger \partial_i U) (U^\dagger \partial_j U) (U^\dagger \partial_k U)] \sim \det(\xi)$.

Due to the symmetry of U , elements of matrix ξ can be decomposed in the following way:

$$\xi_i^a = A \delta_{ia} + B n_i n_a + C \varepsilon_{iak} n_k.$$

Express $\det(\xi)$ in terms of A, B and C . You should get an answer, which is proportional to $(A^2 + C^2)(A + B)$.

In the last step calculate A, B and C . To this end expand U using de'Moivre (or Euler) formula for the exponent. For this you have to prove that $(\vec{n} \cdot \vec{\tau})^2 = 1$.

In order to differentiate U it is useful to use the following identities (prove them!)

$$\begin{aligned} \partial_i r &= n_i, \\ \partial_i r_k &= \frac{1}{r} (\delta_{ik} - n_i n_k). \end{aligned}$$

2. Calculate the field tensor $F_{\mu\nu}$ for the pure gauge $A_\mu = -\frac{i}{g} U \partial_\mu U^\dagger$.

3. Prove that

$$\int_{-T/2}^{T/2} d\tau_1 \int_{\tau_1}^{T/2} d\tau_2 \dots \int_{\tau_{n-1}}^{T/2} d\tau_n = \frac{1}{n!} T^n.$$