QCD
problem set 8

1. Using the result from the previous set calculate the following matrix element:

$$\langle y| \exp\left(-i\frac{\hat{p}^2}{2m}\epsilon\right)|x\rangle$$

where ϵ is a small time lapse. Note that for a free particle ($V=0$) $\epsilon = T$ need not to be small and the above matrix element is in fact the exact propagator K_{free}.

2. Lagrange function for the harmonic oscillator reads:

$$L = \frac{m}{2}\dot{x}(t)^2 - \frac{m\omega^2}{2}x(t)^2.$$

Calculate the classical trajectory leading from point $(x_a, t_a) \to (x_b, t_b)$. Calculate the classical action along this trajectory.

HINT: After finding the classical trajectory $\bar{x}(t)$, calculate the action integrating by parts and using equations of motion.

3. For certain values $\omega(t_b - t_a) = \omega T$ both classical trajectory and classical action exhibit singularities. Find conditions that make them both finite. Discuss meaning of these conditions.

4. Show that quantum contribution to K denoted by F, where

$$K = F(T)\, e^{\frac{i}{\hbar}S[\bar{x}(t)]}$$

reads as follows

$$F(t_b - t_a) = \int [\mathcal{D}y(t)]\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b}\frac{1}{2}m(\dot{y}^2 - \omega^2 y^2)\,dt}\,.$$

Note that the system does not distinguish any specific time, hence the amplitude may depend only on the difference $T = t_b - t_a$.

One of the methods of calculating of calculating F is to expand

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi t}{T}\,, \qquad\qquad n > 0\,.$$

This representation of $y(t)$ satisfies the boundary conditions, $y(0) = y(T) = 0$. Note that

$$\int [\mathcal{D}y(t)] \sim \prod_n da_n$$

with all kinds of factors in front, *but we do not need to calculate them.* This is so because we know the normalization of F in the limit $\omega \to 0$, which is just the free particle propagator from problem 2. Using the fact that functions $\sin \frac{n\pi t}{T}$ form a complete set of orthogonal functions over the time interval $0 \le t \le T$ one can easily compute the argument of the exponent in F, and then perform the Gaussian integrals over da_n's. Final answer can be obtained by means of the following identity (prove it!):

$$\lim_{N\to\infty} \prod_{n=1}^{N} \left(1 - \frac{\omega^2 T^2}{n^2 \pi^2}\right)^{-\frac{1}{2}} = \left(\frac{\sin \omega T}{\omega T}\right)^{-\frac{1}{2}} .$$