

QCD
problem set 4

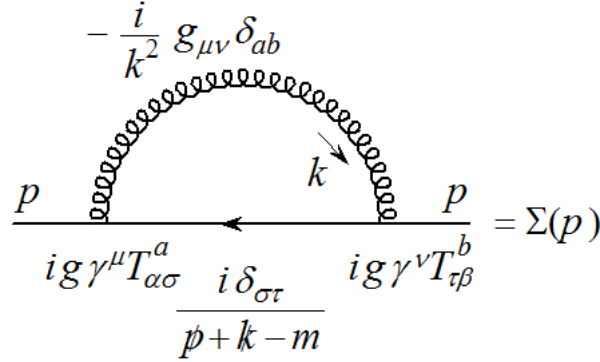


Figure 1: Feynman diagram corresponding to the quark self-energy. Time flow right to left.

In this problem set we shall perform detailed calculation of the fermion self-energy in QCD depicted in Fig. 1. Assume $p^2 \neq 0$. Note that the diagram (1) is almost the same as in QED, except for the color T generators, that enter in the quark-gluon vertices. Gluon propagator is in the Feynman gauge.

The problem is divided into a few steps. During last tutorials we have shown that the mathematical expression $\Sigma(p)$ corresponding to the diagram (1) reads

$$\Sigma(p) = -g^2 \mu^{4-d} C_F \delta_{\alpha\beta} \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu (\not{p} + \not{k}) \gamma_\mu}{(p+k)^2 k^2} \quad (1)$$

where $d = 4 - 2\varepsilon$ is the dimensionality of space-time. Using

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

and

$$g_{\mu\nu} g^{\mu\nu} = d.$$

we have shown that

$$\begin{aligned} \Sigma(p) &= 2(1 - \varepsilon) g^2 \mu^{2\varepsilon} C_F \delta_{\alpha\beta} \int \frac{d^d k}{(2\pi)^d} \frac{\not{p} + \not{k}}{(p+k)^2 k^2} \\ &= 2(1 - \varepsilon) g^2 \mu^{2\varepsilon} C_F \delta_{\alpha\beta} [\not{p} I + \gamma_\mu I^\mu] \end{aligned} \quad (2)$$

where

$$\{I, I^\mu\} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p+k)^2 k^2} \{1, k^\mu\}. \quad (3)$$

Next, we have introduced Feynman parametrization of the propagators

$$\frac{1}{(p+k)^2 k^2} = \int_0^1 dx \frac{1}{(k^2 + 2xpk + xp^2)^2} \quad (4)$$

arriving at

$$\{I, I^\mu\} = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - M^2)^2} \{1, k^\mu - xp^\mu\} \quad (5)$$

where

$$M^2 = -x(1-x)p^2. \quad (6)$$

We have employed change of variables

$$k^\mu \rightarrow k^\mu + xp^\mu. \quad (7)$$

Complete the calculations.

1. In order to calculate the integral over $d^d k$, which is the integral in Minkowski space, we observe that (see Fig. 2):

$$\left\{ \int_{-\infty}^{\infty} + \int_{C_R} + \int_{+i\infty}^{-i\infty} \right\} dk^0 = 0. \quad (8)$$

To see this reinstall Feynman $+i\epsilon$ prescription.

$$k^2 - M^2 \rightarrow k^2 - M^2 + i\epsilon$$

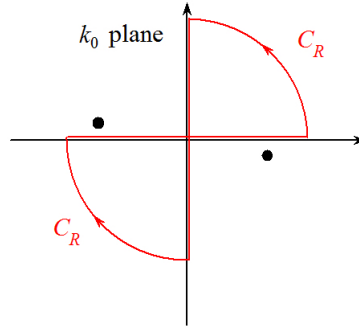


Figure 2: Integration contour over k_0 . Black dots denote poles of Feynman propagators.

Since the integral over C_R vanishes

$$\int_{-\infty}^{\infty} dk^0 = - \int_{+i\infty}^{-i\infty} dk^0 = i \int_{-\infty}^{+\infty} dE \quad (9)$$

where $k^0 = iE$. Therefore the integral over $d^d k$ in Minkowski space transforms into the Euclidean integral

$$\{I, I^m\} = i \int_0^1 dx \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{(-\vec{k}^2 - M^2)^2} \{1, k^m - xp^m\} \quad (10)$$

where

$$k^m = (E, k^1, k^2, \dots, k^{d-1}), \text{ with } m = 1, 2, \dots, d. \quad (11)$$

2. Since nothing depends on the angles, except of k^m , which is nullified by the angular integration, we can use (we shall prove this later, but because full angular integral corresponds to the surface of a sphere of radius $r = 1$ in d dimensions, you can check that the formula below is right for $d = 2$ or 3):

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \quad (12)$$

After the angular integration we arrive at (using $d = 4 - 2\epsilon$):

$$\{I, I^\mu\} = \frac{i}{\Gamma(2 - \epsilon)} \frac{2\pi^{2-\epsilon}}{(2\pi)^{4-2\epsilon}} \int_0^1 dx \{1, -xp^\mu\} \int_0^\infty dk \frac{k^{d-1}}{(k^2 + M^2)^2}.$$

Changing variables to $r = k/M$, and then $t = r^2$, you should get two integrals that are representations of the Euler beta functions:

$$\int_0^\infty dt \frac{t^{x-1}}{(1+t)^{x+y}} = B(x, y),$$

$$\int_0^1 dx x^{\alpha-1} (1-x)^{\beta-1} = B(\alpha, \beta). \quad (13)$$

Identify values of x , y , β and α and then write the final expression for I and I^μ in terms of Euler Γ functions only (use the well known expression for beta functions).

3. Using

$$z\Gamma(z) = \Gamma(z+1), \quad (14)$$

$$\Gamma(1/2) = \sqrt{\pi}. \quad (15)$$

and

$$\Gamma(1 - \epsilon) = \exp\left(\gamma\epsilon + \frac{\pi^2}{12}\epsilon^2 + \dots\right) \quad (16)$$

where γ is Euler constant, you should arrive at the final answer for $\Sigma(p)$:

$$\Sigma(p) = i\psi C_F \frac{\alpha_s}{4\pi} \left(\frac{\mu^2 4\pi e^{-\gamma}}{-p^2} \right)^\varepsilon \left(\frac{1}{\varepsilon} + 1 \right)$$

where

$$\alpha_s = \frac{g^2}{4\pi}. \quad (17)$$

4. In this problem we shall calculate the d -dimensional angular integral given in problem 2. To this end we shall introduce spherical coordinates in d dimensions. First we chose arbitrarily a d -th axis (equivalent of the z axis in three dimensions) and project on it \vec{k} vector with $\cos \theta_{d-1}$. Therefore a projection on the $d-1$ dimensional subspace orthogonal do the d -th axis is $k \sin \theta_{d-1}$. Now we choose an axis in this $d-1$ dimensional subspace, the $d-1$ axis, and project on this axis this projection (*i.e.* $k \sin \theta_{d-1}$) with $\cos \theta_{d-2}$. Next, a projection on the the $d-2$ dimensional subspace orthogonal do the d -th and $d-1$ axes involves $\sin \theta_{d-2}$. We continue this procedure until we "run out of dimensions" with the result:

$$\begin{aligned} k_d &= k \cos \theta_{d-1}, \\ k_{d-1} &= k \sin \theta_{d-1} \cos \theta_{d-2}, \\ &\dots \\ k_2 &= k \sin \theta_{d-1} \sin \theta_{d-2} \dots \cos \theta_1, \\ k_1 &= k \sin \theta_{d-1} \sin \theta_{d-2} \dots \sin \theta_1, \end{aligned} \quad (18)$$

where $\theta_1 \in (0, 2\pi)$, $\theta_{i>1} \in (0, \pi)$. Compute

$$\int d\Omega_d = \int \prod_{i=1}^{d-1} (\sin^{i-1} \theta_i d\theta_i) = 2 \prod_{i=1}^{d-1} \left(\int_0^\pi \sin^{i-1} \theta_i d\theta_i \right) \quad (19)$$

using

$$\int_0^\pi \sin^n \theta d\theta = B \left(\frac{1+n}{2}, \frac{1}{2} \right). \quad (20)$$