QCD
problem set 3

1. Prove that special unitary matrix belonging to SU(N) is parametrized by N^2-1 real parameters. To this end use that
$$U^\dagger U = UU^\dagger = 1, \quad \det U = 1.$$

2. Show that
$$\det U = \exp(\mathrm{Tr}\log U).$$

3. It follows from the above that
$$U = \exp\Big(-i\sum_{a=1}^{N^2-1}\theta_a T^a\Big)$$
where T^a are hermitian, traceless and linearly independent matrices called generators, and θ_a are real parameters. Generators satisfy
$$[T^a, T^b] = i\, f_{abc} T^c \tag{1}$$
where f_{abc} are totally antisymmetric constants. For SU(2) $f_{abc} = \epsilon_{abc}$. Show that any other set of matrices
$$T'^{\,a} = U^\dagger T^a U$$
satisfy the above commutation relations. Convince yourself that
$$[-T^{*\,a}, -T^{*\,b}] = i\, f_{abc}(-T^{*\,c}).$$
For SU(2)
$$-T^{*\,a} = U^\dagger T^a U.$$
Find U.

4. Prove Jacobi identity
$$\big[T^m, [T^n, T^l]\big] + \big[T^n, [T^l, T^m]\big] + \big[T^l, [T^m, T^n]\big] = 0.$$
Show that from the Jacobi identity it follows that matrices
$$\left(T^l_{\mathrm{adj}}\right)_{mn} = -i f_{lmn}$$
satisfy commutation relation (1).

5. A denotes $N^2 - 1$ dimensional vector $A = (a^1, \dots, a^{N^2-1})$. Compute infinitesimal transformation for $\theta_m \to 0$

$$A' = U^{\text{adj}} A \;\to\; a'^m = a^m + ? \tag{2}$$

Here

$$U^{\text{adj}} = \exp\Big(-i \sum_{a=1}^{N^2-1} \theta_a T^a_{\text{adj}} \Big)$$

6. Define matrix $\boldsymbol{A}$ in terms of vector A:

$$\boldsymbol{A} = \sum_{n=1}^{N^2-1} a^n T^n.$$

Show that under infinitesimal transformation

$$\boldsymbol{A}' = U \boldsymbol{A} U^\dagger$$

coordinates of vector A transform as in (2).