QCD lecture 9

November 23

QM - reminder

Schrödinger eq. $$i\hbar \frac{\partial \Psi(x,t_b)}{\partial t_b} = H\Psi(x,t_b)$$

propagates solution from $a=(x_a, t_a)$ to $b=(x_b, t_b)$ $$\Psi(x,t_b) = e^{-\frac{i}{\hbar}H(t_b-t_a)}\Psi(x,t_a)$$

(remember H is an operator)

Define propagator: $$K(b,a) = <x_b|\, e^{-\frac{i}{\hbar}H(t_b-t_a)}\, |x_a>$$

recall Dirac notation $\Psi(x) = <x|\Psi>$ and plane wave solution $<x|p> = N e^{\frac{i}{\hbar}px}$

complex conjugate $<p|x> = N e^{-\frac{i}{\hbar}px}$

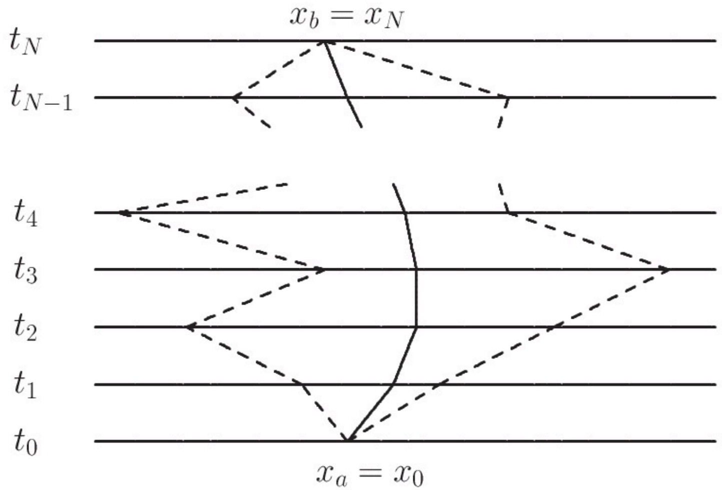
completness relation $$\sum_p |p><p| = \sum_x |x><x| = 1$$

We shall use the following normalization: $$<p|y> = \sqrt{\frac{1}{2\pi\hbar}} e^{-\frac{i}{\hbar}py}$$

Path integral for the propgator

$$K(b,a) = \,<x_b|\, e^{-\frac{i}{\hbar}H(t_b-t_a)}\, |x_a>$$

Discretize time



set $\hbar = m = 1$

"slice" evolution operator

$$e^{-i(t_b-t_b)H} = e^{-i\epsilon N H} = e^{-i\epsilon H}\, e^{-i\epsilon H} \dots e^{-i\epsilon H}$$

insert inbetween unity $1 = \int dx_j\, |x_j><x_j|$

$$<x_b|e^{-i(t_b-t_a)H}|x_a> = \int <x_b|e^{-i\epsilon H}|x_{N-1}>\, dx_{N-1}\, <x_{N-1}|e^{-i\epsilon H}|x_{N-2}>$$

$$\dots\ <x_2|e^{-i\epsilon H}|x_1>\, dx_1\, <x_1|e^{-i\epsilon H}|x_a>$$

Path integral for the propgator

Decompose hamiltonian $$H = \frac{p^2}{2m} + V(x) = K + V$$

and use: $$e^{-i\epsilon H} = e^{-i\epsilon(K+V)} = e^{-i\epsilon K} e^{-i\epsilon V} + O(\epsilon^2)$$

which is true only for small ε

Baker-Cambell-Hausdorff: define C $$e^A e^B = e^C$$

then $$C = A + B + \underbrace{\frac{1}{2}[A,B]}_{\sim \varepsilon^2} + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[[A,B],B] + \ ...$$

Therefore

$$K(b,a) = <x_b|e^{-i(t_b-t_a)H}|x_a>$$

$$= \int <x_b|e^{-i\epsilon K}|x_{N-1}> e^{-i\epsilon V(x_{N-1})}\, dx_{N-1} <x_{N-1}|e^{-i\epsilon K}|x_{N-2}>$$

$$\times\ e^{-i\epsilon V(x_{N-2})} dx_{N-2}\ \ ...\ \ dx_1 <x_1|e^{-i\epsilon K}|x_a> e^{-i\epsilon V(x_a)}$$

Path integral for the propgator

We need to calculate $< x|e^{-\frac{i}{\hbar}\epsilon K}|y> = \int dp <x|e^{-\frac{i}{\hbar}\epsilon\frac{p^2}{2m}}|p><p|y>$

(distinguish operators from eigenalues)

$$= \int dp <x|p> e^{\frac{-i\epsilon p^2}{\hbar 2m}} <p|y>$$

recall normalization $<p|y> = \sqrt{\frac{1}{2\pi\hbar}} e^{-\frac{i}{\hbar}py}$

$$<x|e^{-\frac{i}{\hbar}\epsilon K}|y> = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dp\, e^{\frac{-i\epsilon p^2}{2m\hbar}} e^{-\frac{i}{\hbar}(y-x)p} = \sqrt{\frac{m}{2i\pi\hbar\epsilon}}\, e^{im\frac{(y-x)^2}{2\epsilon\hbar}}$$

where we have used

$$\int_{-\infty}^{+\infty} dx\, e^{ax^2+bx} = \sqrt{\frac{\pi}{-a}} e^{-\frac{b^2}{4a}}, \qquad \mathrm{Re}\, a \le 0$$

$$\frac{i}{2\hbar} m \left(\frac{y-x}{\epsilon}\right)^2 \epsilon = \frac{i}{\hbar}\frac{mv^2}{2}\epsilon$$

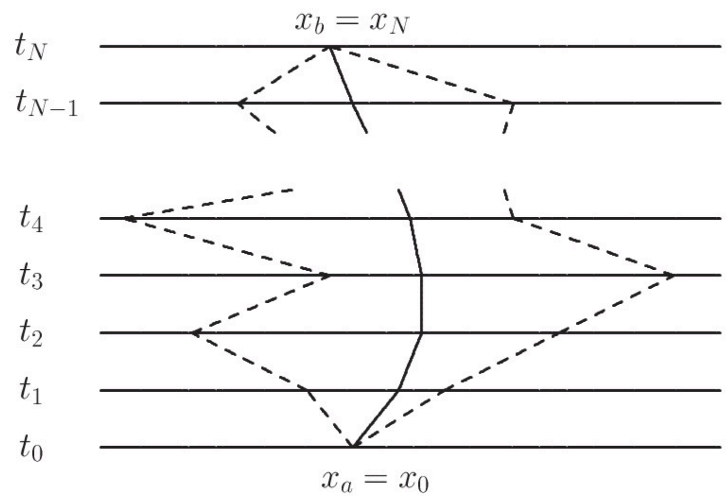
bur remember:

$$L_j = \tfrac{1}{2}m\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2 - V(x_j)$$

Path integral for the propgator

$$K(b,a) = \lim_{\epsilon\to 0} \sqrt{\frac{m}{2i\epsilon\hbar\pi}} \int \prod_{j=1}^{N-1} dx_j \sqrt{\frac{m}{2i\epsilon\hbar\pi}} \prod_{k=0}^{N-1} e^{\frac{i}{\hbar}\epsilon L_k}$$

$$\stackrel{\text{def}}{=} \int [\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt L[x(t),\dot{x}(t)]}$$



Define functional integration measure integration over all traiectories from a to b

$$[\mathcal{D}x(t)] = dx_1 \ldots dx_{N-1} \left(\frac{m}{2i\epsilon\hbar\pi}\right)^{\frac{1}{2}N}$$

and use definition of action

$$\lim_{\epsilon\to 0} \sum_{j=0}^{N-1} \epsilon L_j = \int_{t_a}^{t_b} dt\, L(x(t),\dot{x}(t)) = S[x(t)]$$

to arrive at

$$K(b,a) = \int [\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}S[x(t)]}$$

special role of the classical trajectory
i.e. stationary point of action

Euclidean path integral

Change $t \to -i\tau$

then $K(x_b, x_a, -i\tau) = <x_b|e^{-\frac{H}{\hbar}\tau}|x_a>$

$$= \sum_{n,n'} <x_b|E_n><E_n|\, e^{-\frac{H}{\hbar}\tau}|E_{n'}><E_{n'}|x_a>$$

$$= \sum_n e^{-\frac{E_n}{\hbar}\tau}\phi_n(x_b)\phi_n^*(x_a)\,.$$

for large τ only the ground state survives

Feynman-Kac formula $E_0 = -\lim_{\tau\to\infty}\left\{\frac{\hbar}{\tau}\ln\left(K(x_b, x_a, -i\tau)\right)\right\}$

In Euclidean one can perform computer simulations

$$K_n(x, x, -i\tau) = \int dx_1 dx_2 \ldots dx_n \left(\frac{1}{2\pi\epsilon}\right)^{\frac{1}{2}(n+1)} e^{-\epsilon\sum_{j=0}^{n}\left\{\frac{1}{2}\left(\frac{x_{j+1}-x_j}{\epsilon}\right)^2 + V(x_j)\right\}}$$

Gaussian functional integrals

Assume that path integral is the way we formulate QM (and QFT). All properties and equations are derived from the path integral. In practice we deal with Gaussian functional integrals:

$$L(\dot{x},x,t) = a(t)\,\dot{x}^2(t) + b(t)\,\dot{x}x + c(t)\,x^2 + d(t)\dot{x} + e(t)\,x + f(t)$$

Propagator:

$$K(x_b,x_a,t_b-t_a) = \int [\mathcal{D}x(t)]\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} dt L(\dot{x},x,t)}$$

To evaluate K decompose the quantal trajectory into the classicel one $\bar{x}(t)$

$$\delta S[x(t)] = 0 \qquad \text{gives} \qquad \bar{x}(t)$$

and a fluctuation $y(t)$:

$$x(t) = \bar{x}(t) + y(t)\,, \qquad y(t_b) = y(t_a) = 0$$

Since terms linear in y vanish

$$S[\bar{x}(t)+y(t)] = S[\bar{x}(t)] + \tfrac{1}{2}\delta^2 S[y(t)]$$

$$= S[\bar{x}] + \frac{1}{2!}\int_0^T \left[\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y}^2 + 2\frac{\partial^2 L}{\partial x \partial \dot{x}}\dot{y}y + \frac{\partial^2 L}{\partial x^2}y^2\right] dt$$

for convenience $T = t_b$ - t_a

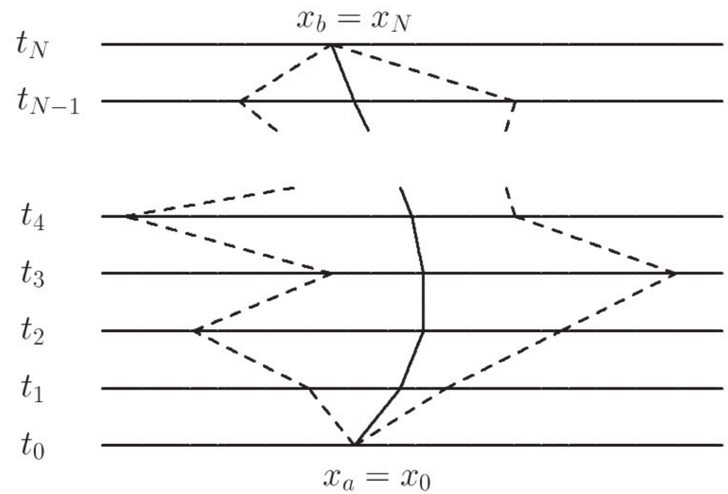
Gaussian functional integrals

Since $\bar{x}(t)$ is fixed we have $\mathcal{D}x(t) = \mathcal{D}y(t)$
and

$$K(x_b, x_a, t_b - t_a) = F(t_b - t_a)\, e^{\frac{i}{\hbar} S[\bar{x}(t)]}$$

where

$$F(t_b - t_a) = \int [\mathcal{D}y(t)]\, e^{\frac{i}{\hbar}\frac{1}{2}\delta^2 S[y(t)]}$$



Recall:

$$\delta^2 S = \int_0^T \left[\dot{y}\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y} + 2y\frac{\partial^2 L}{\partial x \partial \dot{x}}\dot{y} + y\frac{\partial^2 L}{\partial x^2}y\right] dt$$

identities:
(integration by parts)

$$\dot{y}\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y} = \frac{d}{dt}\left(y\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y}\right) - y\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y}\right)$$

$$y(0) = y(T) = 0$$

$$2y\frac{\partial^2 L}{\partial x \partial \dot{x}}\dot{y} = \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}}y^2\right) - y\frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}}\right)y$$

we get
definition of D

$$\delta^2 S = -\int_0^T y\left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y}\right) + \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}}\right)y - \frac{\partial^2 L}{\partial x^2}y\right] dt = \int_0^T y\, D(t) y\, dt$$

Gaussian functional integrals

$$\delta^2 S \;=\; -\int_0^T y\left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y}\right) + \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}}\right) y - \frac{\partial^2 L}{\partial x^2} y\right] dt \;=\; \int_0^T y\, D(t) y\, dt\,.$$

D is a Sturm-Liouville operator $\quad D(t) y_n(t) \;=\; \lambda_n y_n(t)\,, \quad n = 1, 2, 3, \ldots\,, \quad \lambda_1 < \lambda_2 < \ldots$

Example: $\quad L = \frac{1}{2} m \dot{x}^2 - V(x) \qquad D(t) = -m\frac{\partial^2}{\partial t^2} - \frac{\partial^2 V}{\partial x^2}\bigg|_{x=\bar{x}(t)}$

Use y_n basis to expand $\quad y(t) = \sum_{n=1}^{\infty} a_n\, y_n(t) \quad$ then $\quad \delta^2 S[y] = \sum_{n=1}^{\infty} \lambda_n a_n^2$

and $\quad [\mathcal{D}y(t)] \sim \prod_{n=1}^{\infty} da_n$

$$F(T) \sim \prod_{n=1}^{\infty} da_n \, \exp\left(\frac{i}{2\hbar}\lambda_n a_n^2\right) \sim \sqrt{\frac{1}{\prod_n \lambda_n}} = \sqrt{\frac{1}{\det D(t)}}$$

Path integral revisited

We have performed dp integral using a specific form of the hamiltonian

$$H = \frac{p^2}{2m} + V(x)$$

however we do need to use this information. We only have to remember

$$\mathcal{L} = p\dot{q} - H(p,q)$$

Let's recalculate

$$\begin{aligned}
\langle x, t+\varepsilon | \, y, t\rangle &= \langle x | \, e^{-iH\varepsilon} \, | y\rangle \\
&= \int \frac{dp}{2\pi} e^{ip(x-y)} e^{-iH\varepsilon} \\
&= \int \frac{dp}{2\pi} \exp i \left[p\frac{(x-y)}{\varepsilon} - H(p,x) \right] \varepsilon \\
&= \int \frac{dp}{2\pi} \exp i \left[p\dot{x} - H(p,x) \right] \varepsilon
\end{aligned}$$

Hence:

$$K(b,a) \sim \int \mathcal{D}[x(t)] \int \mathcal{D}[p(t)] \exp\left(\frac{i}{\hbar} \int dt \, \left[p\dot{x} - H(p,x) \right] \right)$$

Transition amplitudes

Consider matrix element of a position operator Q measuring expectation value of the position at time t_1

$$\langle q_f | e^{-i\mathcal{H}(t_f - t_1)} \, Q \, e^{-i\mathcal{H}(t_1 - t_i)} | q_i \rangle$$

We have

$$Q \quad \to \quad \int dq\, dq' \, |q\rangle \underbrace{\langle q | Q | q' \rangle}_{q\, \delta(q - q')} \langle q' | = \int dq \; q \; |q\rangle\langle q|$$

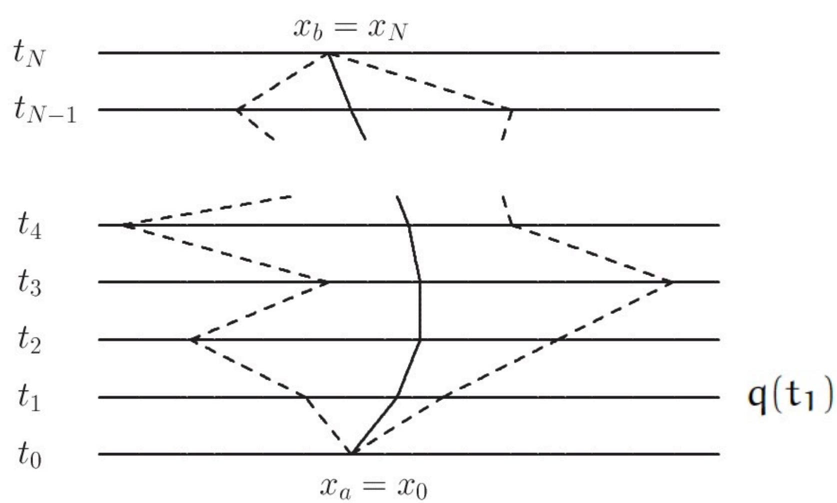
which lead to

$$\langle q_f | e^{-i\mathcal{H}(t_f - t_1)} \, Q \, e^{-i\mathcal{H}(t_1 - t_i)} | q_i \rangle = \int\limits_{\substack{q(t_i) = q_i \\ q(t_f) = q_f}} [Dq(t)] \; q(t_1) \; e^{i\mathcal{S}[q(t)]}$$

Similarly for $t_2 > t_1$

$$\langle q_f | e^{-i\mathcal{H}(t_f - t_2)} \, Q \, e^{-i\mathcal{H}(t_2 - t_1)} \, Q \, e^{-i\mathcal{H}(t_1 - t_i)} | q_i \rangle =$$

$$= \int\limits_{\substack{q(t_i) = q_i \\ q(t_f) = q_f}} [Dq(t)] \; q(t_1) \, q(t_2) \; e^{i\mathcal{S}[q(t)]}$$



Transition amplitudes

F. Gelis: A Stroll Through Field Theory

Define time dependent operator $Q(t) \equiv e^{i\mathcal{H}t}\, Q\, e^{-i\mathcal{H}t}$ and $|q,t\rangle \equiv e^{i\mathcal{H}t}\, |q\rangle$

then

$$\langle q_f, t_f | Q(t_2) Q(t_1) | q_i, t_i \rangle \underset{t_2 > t_1}{=} \int\limits_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)]\; q(t_1)\, q(t_2)\; e^{i\mathcal{S}[q(t)]}$$

operators functions

Note that l.h.s is very different when $t_1 > t_2$, whereas r.h.s. is the same because classical trajectories commute. Introduce time ordering T

$$\mathrm{T}(Q(t_1)Q(t_2)) = \theta(t_1 - t_2)Q(t_1)Q(t_2) + \theta(t_2 - t_1)Q(t_2)Q(t_1)$$

then

$$\langle q_f, t_f | \mathrm{T}\left(Q(t_1) Q(t_2)\right) | q_i, t_i \rangle = \int\limits_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)]\; q(t_1)\, q(t_2)\; e^{i\mathcal{S}[q(t)]}$$

generally

$$\langle q_f, t_f | \mathrm{T}\left(Q(t_1) \cdots Q(t_n)\right) | q_i, t_i \rangle = \int\limits_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)]\; q(t_1) \cdots q(t_n)\; e^{i\mathcal{S}[q(t)]}$$

Functional sources an derivatives

One can derive transtion amplitudes with the help of generating functional

$$Z_{fi}[j(t)] \equiv \langle q_f, t_f | T \exp i \int_{t_i}^{t_f} dt\, j(t)\, Q(t) | q_i, t_i \rangle$$

where $j(t)$ is some arbitrary function of time and $Q(t)$ is and operator
Amplidudes are given as functional derivatives

$$\langle q_f, t_f | T\left(Q(t_1) \cdots Q(t_n)\right) | q_i, t_i \rangle = \left. \frac{\delta^n Z_{fi}[j]}{i^n \delta j(t_1) \cdots \delta j(t_n)} \right|_{j \equiv 0}$$

Functional derivatives act essentially as regular differenciation with one additional property

$$\frac{\delta j(t)}{\delta j(t')} = \delta(t - t')$$

values of function $j(t)$ at different times are independent variables

Generating functional has path integral representation (Lagrange)

$$Z_{fi}[j(t)] = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dq(t)]\; e^{iS[q(t)] + i\int_{t_i}^{t_f} dt\, j(t) q(t)}$$

Functional sources an derivatives

One can derive transtion amplitudes with the help of generating functional

$$Z_{fi}[j(t)] \equiv \langle q_f, t_f | T \exp i \int_{t_i}^{t_f} dt\, j(t)\, Q(t) | q_i, t_i \rangle$$

where $j(t)$ is some arbitrary function of time and $Q(t)$ is and operator
Amplidudes are given as functional derivatives

$$\langle q_f, t_f | T\left(Q(t_1)\cdots Q(t_n)\right) | q_i, t_i \rangle = \left. \frac{\delta^n\, Z_{fi}[j]}{i^n \delta j(t_1)\cdots \delta j(t_n)} \right|_{j\equiv 0}$$

Functinal derivatives act essentially as regular differenciation with one additional property

$$\frac{\delta j(t)}{\delta j(t')} = \delta(t - t')$$

values of function $j(t)$ at different times are independent variables

Generating functional has path integral representation (Hamilton)

$$Z_{fi}[j(t)] = \int\limits_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} [Dp(t)Dq(t)] \times \exp\left\{ i \int_{t_i}^{t_f} dt\, \left(p(t)\dot{q}(t) - \mathcal{H}(p(t), q(t)) + j(t)q(t)\right) \right\}$$

Ground state projection

Initial and final states do not have to be position eigenstates. Consider some operator O and some state ψ

$$\psi(q) \equiv \langle q|\psi\rangle$$

Then

$$\langle \psi_f, t_f|\mathcal{O}|\psi_i, t_i\rangle = \int dq_i dq_f \, \psi_f^*(q_f)\, \psi_i(q_i)\, \langle q_f, t_f|\mathcal{O}|q_i, t_i\rangle$$

In practice we often need matrix element when ininitial and final states are the ground states:

$$\begin{aligned} |q_i, t_i\rangle &= e^{i\mathcal{H}t_i}|q_i\rangle \\ &= \sum_{n=0}^{\infty} e^{i\mathcal{H}t_i}|n\rangle\langle n|q_i\rangle \\ &= \sum_{n=0}^{\infty} \psi_n^*(q_i)\, e^{iE_n t_i}|n\rangle \end{aligned}$$

Assume that $E_0 = 0$ (shifting energy) and multiply the hamiltonian by $1 - i0^+$

Then all factors $\exp(i(1 - i0^+)E_n t_i)$ go to 0 for $t_i \to -\infty$ except for the ground state

Ground state projection

With $1 - i0^+$ prescripton

$$\lim_{t_i \to -\infty} |q_i, t_i\rangle = \psi_0^*(q_i)\, |0\rangle \qquad \lim_{t_f \to +\infty} \langle q_f, t_f| = \psi_0(q_f)\, \langle 0|$$

The generating functional is then vacuum expectation value and reads (Hamilton)

$$Z[j(t)] = \int [Dp(t)Dq(t)] \times \exp\left\{i \int dt\, \left(p(t)\dot{q}(t) - \underline{(1 - i0^+)}\mathcal{H}(p(t), q(t)) + j(t)q(t)\right)\right\}$$

or (Lagrange)

$$Z[j(t)] = \int [Dq(t)] \times \exp\left\{i \int dt\, \left((1 + i0^+)\frac{m\dot{q}^2(t)}{2} - (1 - i0^+)V(q(t)) + j(t)q(t)\right)\right\}$$

Normalization $Z[0] = 1$