QCD lecture 8

November 18

Axial anomaly

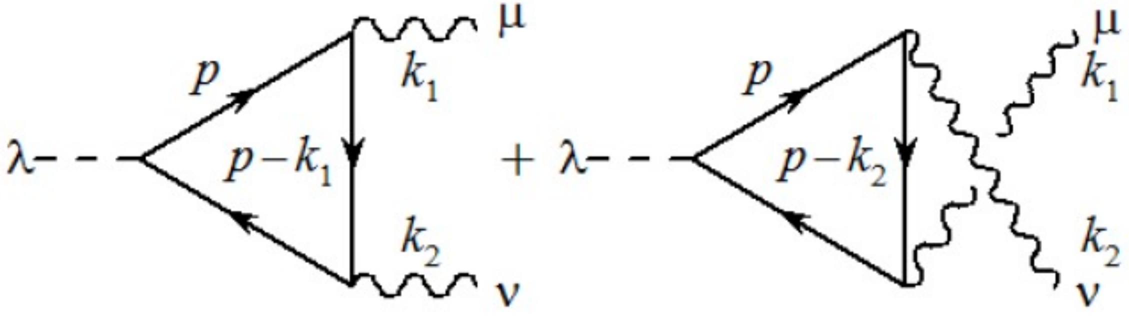
pseudoscalar density

Gauge invariance of QED (and QCD): $q_\mu j^\mu(q) = \bar{u}(p')\gamma^\mu u(p) = 0$

divergence of axial-vector current: $q_\mu j_5^\mu(q) = \bar{u}(p')\gamma^\mu\gamma_5 u(p) = 2m\,\bar{u}(p')\gamma_5 u(p)$

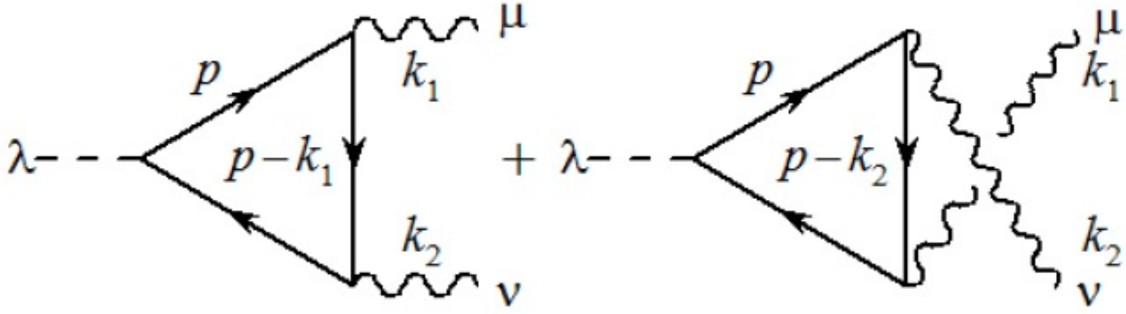
Axial current is conserved for massless fermions: chiral symmetry

It is not possible to maintain both symmetries when loop corrections are included. This is called: AXIAL ANOMALY



photons are bosons and they are not distinguishable hence amplitude has to be symmetrized

Naïve current conservation



$$q = k_1 + k_2$$

Skipping coupling constants (charges) the amplitude reads:

$$\begin{aligned} T_{\mu\nu\lambda} \;=\; & -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\nu\frac{i}{(\not p - \not k_1) - m}\gamma_\mu\right] \\ & -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\mu\frac{i}{(\not p - \not k_2) - m}\gamma_\nu\right] \end{aligned}$$

Naively we expect:

$$k_1^\mu T_{\mu\nu\lambda} = k_2^\nu T_{\mu\nu\lambda} = 0 \qquad q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu}$$

Naïve current conservation

Vector current, first diagram:

$$k_1^\mu T_{\mu\nu\lambda} > \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\not{k}_1 \frac{i}{\not{p}-m}\right]$$

use trick:

$$\not{k}_1 = (\not{p}-m) - ((\not{p}-\not{k}_1)-m)$$

we get:

$$= i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\right] - i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{\not{p}-m}\right]$$

Naïve current conservation

Vector current, first diagram:

$$k_1^\mu T_{\mu\nu\lambda} > \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\not k_1 \frac{i}{\not p - m}\right]$$

use trick:

$$\not k_1 = (\not p - m) - ((\not p - \not k_1) - m)$$

we get:

$$= i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\right] - i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\nu \frac{i}{\not p - m}\right]$$

same trick with the second diagram gives

$$= i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\nu \frac{i}{\not p - m}\right] - i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not k_2) - m}\gamma_\nu \frac{i}{\not p - m}\right]$$

Naïve current conservation

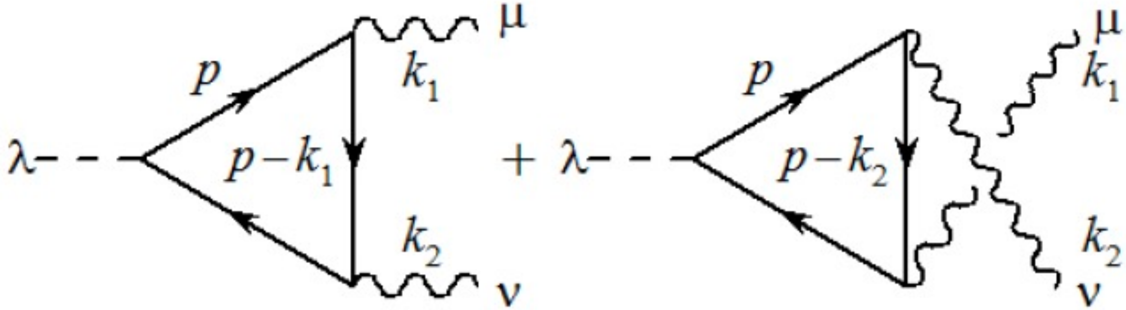
$$k_1^\mu T_{\mu\nu\lambda} \sim \int \frac{d^4 p}{(2\pi)^4}$$

$$\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{i}{(\not{p}-\not{q})-m} \gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{i}{(\not{p}-\not{k}_2)-m} \gamma_\nu \frac{i}{\not{p}-m}\right] \right\}$$

change variable in the first integral $p \to p + k_1$

It seems we get zero

Naïve current conservation



$$q = k_1 + k_2$$

Skipping coupling constants (charges) the amplitude reads:

$$\begin{aligned} T_{\mu\nu\lambda} \;=\; & -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p-\not q)-m}\gamma_\nu\frac{i}{(\not p-\not k_1)-m}\gamma_\mu\right] \\ & -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p-\not q)-m}\gamma_\mu\frac{i}{(\not p-\not k_2)-m}\gamma_\nu\right] \end{aligned}$$

Naively we expect:

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu}$$

Axial current

To calculate $q^\lambda T_{\mu\nu\lambda}$

we use the following trick:

$$\begin{aligned} \not{q}\gamma_5 &= -\gamma_5 \not{q} \\ &= \gamma_5 \left[(\not{p} - \not{q}) - m \right] - \gamma_5 \left[\not{p} - m \right] \\ &= \gamma_5 \left[(\not{p} - \not{q}) - m \right] + \left[\not{p} - m \right] \gamma_5 + 2m\gamma_5 \end{aligned}$$

and for the first diagram we obtain

$$\begin{aligned} q^\lambda \left[\frac{i}{\not{p} - m} \gamma_\lambda \gamma_5 \frac{i}{(\not{p} - \not{q}) - m} \right] &= 2m \frac{i}{\not{p} - m} \gamma_5 \frac{i}{(\not{p} - \not{q}) - m} \\ &+ i \frac{i}{\not{p} - m} \gamma_5 + i\gamma_5 \frac{i}{(\not{p} - \not{q}) - m} \end{aligned}$$

Axial current

Sum from the two diagrams $$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}$$

$$\begin{aligned}
&\Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \\
= &\int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu\frac{i}{(\not p - \not k_1) - m}\gamma_\mu + \gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\nu\frac{i}{(\not p - \not k_1) - m}\gamma_\mu\right] \\
+ &\int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\mu\frac{i}{(\not p - \not k_2) - m}\gamma_\nu + \gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\mu\frac{i}{(\not p - \not k_2) - m}\gamma_\nu\right]
\end{aligned}$$

Axial current

$$\Delta^{(1)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\gamma_\mu - \frac{i}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not q) - m}\gamma_\mu\right]$$

$$\Delta^{(2)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\mu \frac{i}{(\not p - \not k_2) - m}\gamma_\nu - \frac{i}{(\not p - \not k_1) - m}\gamma_5\gamma_\mu \frac{i}{(\not p - \not q) - m}\gamma_\nu\right]$$

The question is: are $\Delta^{(1,2)}_{\mu\nu}$ equal zero?

Changing variables $\quad p \to p + k_2$

seems to nullify $\Delta^{(1,2)}_{\mu\nu}$. $\quad p \to p + k_1$

However, $\Delta^{(1,2)}_{\mu\nu} \sim \int^{\infty} dp p^3 \frac{1}{p^2} \sim \int^{\infty} dp p$ are UV divergent

Due to the minus sign the divergence is only linear

Mathematical diggression

Consider the integral that is naively zero:

$$\int_{-\infty}^{\infty} dx\,[f(x+a)-f(x)]$$

However, if

$$f(\pm\infty)\neq 0.$$

we can calculate this integral by Taylor expansion:

$$\int_{-\infty}^{\infty} dx\,[f(x+a)-f(x)] = a\,[f(\infty)-f(-\infty)] + \frac{a^2}{2}\,[f'(\infty)-f'(-\infty)] + \ldots$$

it may happen that $\neq 0$

Mathematical diggression

Consider Euclidean integral: $$\Delta(\vec{a}) = \int d^n\vec{r}\,[f(\vec{r}+\vec{a}) - f(\vec{r})]$$

expand in a $$= \int d^n\vec{r}\,\vec{a}\cdot\vec{\nabla} f(\vec{r}) + \dots$$

apply Gauss theorem $$= \vec{a}\cdot\vec{n}\,S_n(R)\,f(\vec{R})$$

where $\vec{n} = \frac{\vec{R}}{R}$ and $S_n(R)$ is a surface of the n sphere, R is regulator.

For even n

$$S_n(R) = \frac{2\pi^{n/2}}{(n/2-1)!}R^{n-1} = \begin{cases} 2\pi R & \text{for} \quad n=2 \\ 2\pi^2 R^3 & \text{for} \quad n=4 \end{cases}$$

In Minkowski space

$$\Delta(a) = 2i\pi^2 a^\mu \lim_{R\to\infty} R^2 R_\mu\, f(R)$$

Shift in full amplitude

$$T_{\mu\nu\lambda} = -i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p-\not q)-m}\gamma_\nu\frac{i}{(\not p-\not k_1)-m}\gamma_\mu\right]$$
$$-i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p-\not q)-m}\gamma_\mu\frac{i}{(\not p-\not k_2)-m}\gamma_\nu\right]$$

define shift vector
and amplitude difference:

$$a = \alpha k_1 + (\alpha-\beta)k_2$$

$$\Delta_{\mu\nu\lambda}(a) = T_{\mu\nu\lambda}(p\to p+a) - T_{\mu\nu\lambda}$$

Strategy:

$$\begin{aligned} q^\lambda T_{\mu\nu l}(a) &= q^\lambda\left(T_{\mu\nu l}(a) - T_{\mu\nu l}(0)\right) + q^\lambda T_{\mu\nu l}(0) \\ &= q^\lambda\Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \end{aligned}$$

$$k_1^\mu T_{\mu\nu l}(a) = k_1^\mu\left(T_{\mu\nu l}(a) - T_{\mu\nu l}(0)\right) + k_1^\mu T_{\mu\nu l}(0)$$

chose a in a way that vector current is conserved
and see what comes out for the axial current

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \left\{ \mathrm{Tr}\left[\frac{1}{\not p + \not a - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p + \not a - \not q) - m}\gamma_\nu \frac{1}{(\not p + \not a - \not k_1) - m}\gamma_\mu \right] - \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p - \not q) - m}\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu \right] \right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \left\{ \mathrm{Tr}\left[\frac{1}{\not p + \not a - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p + \not a - \not q) - m}\gamma_\nu \frac{1}{(\not p + \not a - \not k_1) - m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p - \not q) - m}\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right]\right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p - \not q) - m}\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right] + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Shift in full amplitude

Calculate

(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}\left\{\mathrm{Tr}\left[\frac{1}{\not p+\not a-m}\gamma_\lambda\gamma_5\frac{1}{(\not p+\not a-\not q)-m}\gamma_\nu\frac{1}{(\not p+\not a-\not k_1)-m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not p-m}\gamma_\lambda\gamma_5\frac{1}{(\not p-\not q)-m}\gamma_\nu\frac{1}{(\not p-\not k_1)-m}\gamma_\mu\right]\right\} + (\mu\longleftrightarrow\nu, k_1\leftrightarrow k_2)\,.$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}a^\sigma\frac{\partial}{\partial p^\sigma}\mathrm{Tr}\left[\frac{1}{\not p-m}\gamma_\lambda\gamma_5\frac{1}{(\not p-\not q)-m}\gamma_\nu\frac{1}{(\not p-\not k_1)-m}\gamma_\mu\right] + (\mu\longleftrightarrow\nu, k_1\leftrightarrow k_2)\,.$$

large p limit

$$\frac{1}{p^6}\mathrm{Tr}\left[\not p\gamma_\lambda\gamma_5\not p\gamma_\nu\not p\gamma_\mu\right]$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \left\{ \mathrm{Tr}\left[\frac{1}{\not p + \not a - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p + \not a - \not q) - m}\gamma_\nu \frac{1}{(\not p + \not a - \not k_1) - m}\gamma_\mu \right] - \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p - \not q) - m}\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu \right] \right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2) .$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p - \not q) - m}\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu \right] + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2) .$$

large p limit

$$\frac{1}{p^6}\mathrm{Tr}\left[\not p \gamma_\lambda\gamma_5 \not p \gamma_\nu \not p \gamma_\mu\right]$$

go to Euclidean
apply Gauss th.
$r_0 \to i r_0$
$d^4 r = i d^4 \vec{r}$

$$\Delta_{\mu\nu\lambda}(a) = -\frac{i}{(2\pi)^4} 2\pi^2 a^\sigma \lim_{P\to\infty} P^3 \frac{P_\sigma}{P} \mathrm{Tr}\left[\not P \gamma_\lambda\gamma_5 \not P \gamma_\nu \not P \gamma_\mu\right] \frac{1}{P^6} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \left\{ \mathrm{Tr}\left[\frac{1}{\not p + \not a - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p + \not a - \not q) - m}\gamma_\nu \frac{1}{(\not p + \not a - \not k_1) - m}\gamma_\mu \right] - \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p - \not q) - m}\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu \right] \right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p - \not q) - m}\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu \right] + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

large p limit

$$\frac{1}{p^6}\mathrm{Tr}\left[\not p\gamma_\lambda\gamma_5\not p\gamma_\nu\not p\gamma_\mu\right]$$

go to Euclidean
apply Gauss th.

$$\Delta_{\mu\nu\lambda}(a) = -\frac{i}{(2\pi)^4} 2\pi^2 a^\sigma \lim_{P\to\infty} P^3 \frac{P_\sigma}{P} \mathrm{Tr}\left[\not P\gamma_\lambda\gamma_5\not P\gamma_\nu\not P\gamma_\mu\right]\frac{1}{P^6} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

calculate Trace

$$\mathrm{Tr}\left[\not P\gamma_\lambda\gamma_5\not P\gamma_\nu\not P\gamma_\mu\right] = 4iP^2\varepsilon_{\alpha\mu\nu\lambda}P^\alpha$$

Remember that $\varepsilon_{\alpha\mu\nu\lambda} = -\varepsilon^{\alpha\mu\nu\lambda}$

Shift in full amplitude

We arrive at:

$$\Delta_{\mu\nu\lambda}(a) = \frac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha}\, a_\sigma \lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

Shift in full amplitude

We arrive at:
$$\Delta_{\mu\nu\lambda}(a) = \frac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha}\, a_\sigma \lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

take symmetric limit:
$$\lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} = \frac{1}{4} g^{\sigma\alpha}$$

recall: $a = \alpha k_1 + (\alpha - \beta) k_2$

Shift in full amplitude

We arrive at:
$$\Delta_{\mu\nu\lambda}(a) = \frac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha}\, a_\sigma \lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

take symmetric limit:
$$\lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} = \frac{1}{4} g^{\sigma\alpha}$$

recall: $a = \alpha k_1 + (\alpha - \beta) k_2$

Final result:
$$\begin{aligned}\Delta_{\mu\nu\lambda}(a) &= \frac{1}{8\pi^2}\varepsilon_{\alpha\mu\nu\lambda} a^\alpha + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2) \\ &= \frac{1}{8\pi^2}\varepsilon_{\alpha\mu\nu\lambda}\left(\alpha k_1^\alpha + (\alpha-\beta)k_2^\alpha - \alpha k_2^\alpha - (\alpha-\beta)k_1^\alpha\right) \\ &= \frac{\beta}{8\pi^2}\varepsilon_{\alpha\mu\nu\lambda}\,(k_1 - k_2)^\alpha \,.\end{aligned}$$

depends on β, there is an ambiguity, which we have to fix demanding that vector current is conserved.

Axial current, cont.

Recall:

$$\begin{aligned} q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\ &= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \end{aligned}$$

calculated — finite — needs to be computed

Axial current, cont.

Recall:

$$\begin{aligned} q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\ &= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \end{aligned}$$

calculated finite needs to be computed

Let's calculate

$$\Delta^{(1)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\gamma_\mu - \frac{i}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not q) - m}\gamma_\mu\right]$$

Axial current, cont.

Recall:

$$
\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\
&= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2m T_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}
\end{aligned}
$$

calculated finite needs to be computed

Let's calculate

$$
\begin{aligned}
\Delta^{(1)}_{\mu\nu} &= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p} - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m}\gamma_\mu - \frac{i}{(\not{p} - \not{k}_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{q}) - m}\gamma_\mu\right] \\
&= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{1}{(\not{p} - \not{k}_2) - m}\gamma_5\gamma_\nu \frac{1}{(\not{p} - \not{k}_2 - \not{k}_1) - m}\gamma_\mu - \frac{1}{\not{p} - m}\gamma_5\gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m}\gamma_\mu\right]
\end{aligned}
$$

We can use the same trick as previously $p \to p - k_2$ where $a = -k_2$

Axial current, cont.

Recall:

$$\begin{aligned} q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\ &= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \end{aligned}$$

calculated finite needs to be computed

Let's calculate

$$\begin{aligned} \Delta^{(1)}_{\mu\nu} &= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\gamma_\mu - \frac{i}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not q) - m}\gamma_\mu\right] \\ &= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{1}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{1}{(\not p - \not k_2 - \not k_1) - m}\gamma_\mu - \frac{1}{\not p - m}\gamma_5\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right] \end{aligned}$$

We can use the same trick as previously $p \to p - k_2$ where $a = -k_2$

$$\Delta^{(1)}_{\mu\nu} = -\frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho \lim_{P\to\infty} \frac{P_\rho}{P^2} \mathrm{Tr}\left[\not P \gamma_5 \gamma_\nu (\not P - \not k_1)\gamma_\mu\right]$$

keep k_1, because Tr with 2 P's is zero

Axial current, cont.

$$\Delta^{(1)}_{\mu\nu} = -\frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho \lim_{P\to\infty} \frac{P_\rho}{P^2} \operatorname{Tr}\left[\not{P}\gamma_5\gamma_\nu(\not{P} - \not{k}_1)\gamma_\mu\right]$$

We have

$$\begin{aligned}
\Delta^{(1)}_{\mu\nu} &= \frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho k_1^\sigma \lim_{P\to\infty} \frac{P_\rho P^\alpha}{P^2} \operatorname{Tr}\left[\gamma_\alpha\gamma_5\gamma_\nu\gamma_\sigma\gamma_\mu\right] \\
&= \frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho k_1^\sigma \frac{1}{4}(-) \underbrace{\operatorname{Tr}\left[\gamma_5\gamma_\rho\gamma_\nu\gamma_\sigma\gamma_\mu\right]}_{4i\varepsilon_{\rho\nu\sigma\mu}} \\
&= -\frac{1}{8\pi^2} \varepsilon_{\mu\nu\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}$$

We obtain $\Delta^{(2)}_{\mu\nu}$ by $\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2$, hence

$$\Delta^{(1)}_{\mu\nu} = \Delta^{(2)}_{\mu\nu}$$

Axial current, final

$$
\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\
&= 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} + q^\lambda \Delta_{\mu\nu\lambda}(a) \\
&= 2mT_{\mu\nu} - \frac{1}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho + (k_1+k_2)^\lambda \frac{\beta}{8\pi^2}\varepsilon_{\alpha\mu\nu\lambda}\,(k_1-k_2)^\alpha \\
&= 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho
\end{aligned}
$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}
$$

We ned the first piece

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$\begin{aligned} k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho . \end{aligned}$$

We ned the first piece

$$\begin{aligned} k_1^\mu T_{\mu\nu\lambda} &= -\int \frac{d^4p}{(2\pi)^4} \\ & \left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{a}) - m} \gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{k}_2) - m} \gamma_\nu \frac{1}{\not{p} - m}\right]\right\} \end{aligned}$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}
$$

We ned the first piece

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda} &= -\int \frac{d^4 p}{(2\pi)^4} \\
&\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not a) - m} \gamma_\nu \frac{1}{(\not p - \not k_1) - m} \right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2) - m} \gamma_\nu \frac{1}{\not p - m} \right] \right\}
\end{aligned}
$$

$$
\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2 - \not k_1)) - m} \gamma_\nu \frac{1}{(\not p - \not k_1) - m} \right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2) - m} \gamma_\nu \frac{1}{\not p - m} \right] \right\}
$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$\begin{aligned} k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho . \end{aligned}$$

We ned the first piece

$$k_1^\mu T_{\mu\nu\lambda} = -\int \frac{d^4p}{(2\pi)^4}$$

$$\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{q}) - m} \gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{k}_2) - m} \gamma_\nu \frac{1}{\not{p} - m}\right] \right\}$$

$$\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{k}_2 - \not{k}_1)) - m} \gamma_\nu \frac{1}{(\not{p} - \not{k}_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not{p} - \not{k}_2) - m} \gamma_\nu \frac{1}{\not{p} - m}\right] \right\}$$

$$k_1^\mu T_{\mu\nu\lambda} = -\frac{1}{(2\pi)^4} 2i\pi^2 (-) k_1^\sigma \lim_{R\to\infty} \frac{P_\sigma}{P^2} \mathrm{Tr}\left[\gamma_\lambda \gamma_5 (\not{P} - \not{k}_2) \gamma_\nu \not{P}\right]$$

Vector current

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda} &= -\frac{1}{(2\pi)^4} 2i\pi^2 (-) k_1^\sigma \lim_{R\to\infty} \frac{P_\sigma}{P^2} \mathrm{Tr}\left[\gamma_\lambda \gamma_5 (\not{P} - \not{k}_2) \gamma_\nu \not{P}\right] \\
&= -\frac{1}{8\pi^2} i \frac{1}{4} \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \gamma_\rho \gamma_\nu \gamma_\sigma\right] k_1^\sigma k_2^\rho \\
&= \frac{1}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}
$$

Recall

$$
k_1^\mu T_{\mu\nu\lambda}(a) = k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho = \frac{1+\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho
$$

We need to choose $\beta = -1$ to have vector current conserved!

Axial anomaly

Summarizing:

$$q^{\lambda} T_{\mu\nu\lambda}(a) \quad = \quad 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^{\sigma} k_2^{\rho}$$

$$k_1^{\mu} T_{\mu\nu\lambda}(a) = \frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho} k_1^{\sigma} k_2^{\rho}.$$

Choose $\beta = -1$

$$q^{\lambda} T_{\mu\nu\lambda} = 2mT_{\mu\nu} - \frac{1}{2\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^{\sigma} k_2^{\rho}$$

Axial current is anomalous

This can be translated to the configurations space

$$\partial^{\lambda} J_{\lambda}^{5}(x) = \frac{1}{(4\pi)^2}\varepsilon_{\mu\nu\sigma\rho} F^{\mu\nu}(x) F^{\sigma\rho}(x) + \mathcal{O}(m)$$

Axial anomaly

Summarizing:

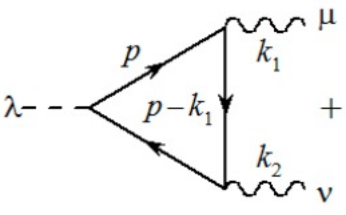
$$q^\lambda T_{\mu\nu\lambda}(a) = 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}k_1^\sigma k_2^\rho$$

$$k_1^\mu T_{\mu\nu\lambda}(a) = \frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho}k_1^\sigma k_2^\rho.$$

- Anomaly is mass independent
- Adler-Bardeen theorem (69): no higher order correctoons
- name: Adler-Bardeen-Jackiw anomaly
- Fujikawa (79) path integral formulation
- In non-Abelian case one can nullify anomaly Tr(…)=0

Choose $\beta = -1$

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} - \frac{1}{2\pi^2}\varepsilon_{\mu\nu\sigma\rho}k_1^\sigma k_2^\rho$$



Axial current is anomalous

This can be translated to the configurations space

$$\partial^\lambda J_\lambda^5(x) = \frac{1}{(4\pi)^2}\varepsilon_{\mu\nu\sigma\rho}F^{\mu\nu}(x)F^{\sigma\rho}(x) + \mathcal{O}(m)$$