QCD lecture 7

November 16

Chiral symmetry

Dirac equation in chrial representation for gamma matrices

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \; \gamma_5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

splits into two equations

$$(i\partial_t - i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\psi_L - m\psi_R = 0, \qquad (i\partial_t + i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\psi_R - m\psi_L = 0,$$

where $\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$. Note that for massless fermions these eqs. are *independent*.

Projection operators: $P_L = \frac{1}{2}(1-\gamma_5),\; P_R = \frac{1}{2}(1+\gamma_5)$ project solutions of

given chirality (eigen value of γ_5)

$$\psi_- = \begin{bmatrix} \psi_L \\ 0 \end{bmatrix}, \quad \psi_+ = \begin{bmatrix} 0 \\ \psi_R \end{bmatrix}$$

Helicity

Helicity: projecton of spin on the particle's momentum:

$$h = \frac{2}{p}\boldsymbol{p}\cdot\boldsymbol{\Sigma} = \frac{1}{p}\begin{bmatrix} \boldsymbol{p}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{p}\cdot\boldsymbol{\sigma} \end{bmatrix} \qquad p = |\boldsymbol{p}|$$

Massless Dirac equation: $$\left(\gamma^0 E - \boldsymbol{\gamma}\cdot\boldsymbol{p}\right)\psi_\pm = 0 \;\rightarrow\; \frac{\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p}}{E}\psi_\pm = \psi_\pm$$

It is easy to show that in the chiral representation for gamm matrices

$$\gamma_5 h = \pm\frac{\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p}}{E_\pm} = \frac{1}{p}\begin{bmatrix} -\boldsymbol{p}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{p}\cdot\boldsymbol{\sigma} \end{bmatrix} \qquad E_\pm = \pm p$$

Here positive energy solutions correspond to particles, and negative ones to antiparticles.

Particles: $\gamma_5 h\psi_\pm = \psi_\pm \;\rightarrow\; h\psi_\pm = \pm\psi_\pm$ helicity = chirality

Antiparticles: $-\gamma_5 h\psi_\pm = \psi_\pm \;\rightarrow\; h\psi_\pm = \mp\psi_\pm$ helicity = - chirality

Axial anomaly

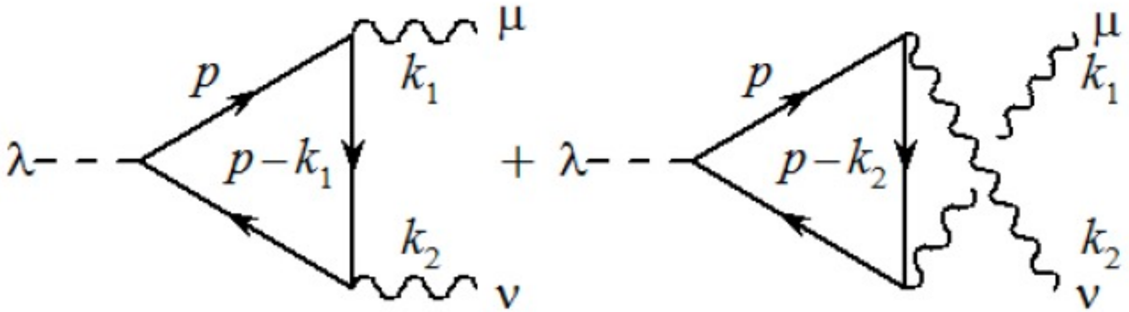
pseudoscalar density

Gauge invariance of QED (and QCD): $q_\mu j^\mu(q) = \bar{u}(p')\gamma^\mu u(p) = 0$

divergence of axial-vector current: $q_\mu j_5^\mu(q) = \bar{u}(p')\gamma^\mu\gamma_5 u(p) = 2m\,\bar{u}(p')\gamma_5 u(p)$

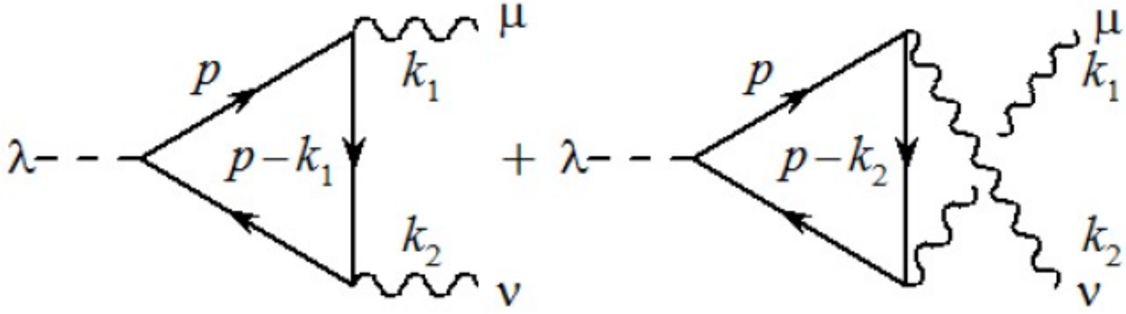
Axial current is conserved for massless fermions: chiral symmetry

It is not possible to maintain both symmetries when loop corrections are included. This is called: AXIAL ANOMALY



photons are bosons and they are not distinguishable hence amplitude has to be symmetrized

Naïve current conservation



$$q = k_1 + k_2$$

Skipping coupling constants (charges) the amplitude reads:

$$\begin{aligned} T_{\mu\nu\lambda} &= -i \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} \left[\frac{i}{\not p - m} \gamma_\lambda \gamma_5 \frac{i}{(\not p - \not q) - m} \gamma_\nu \frac{i}{(\not p - \not k_1) - m} \gamma_\mu \right] \\ & \quad -i \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr} \left[\frac{i}{\not p - m} \gamma_\lambda \gamma_5 \frac{i}{(\not p - \not q) - m} \gamma_\mu \frac{i}{(\not p - \not k_2) - m} \gamma_\nu \right] \end{aligned}$$

Naively we expect:

$$k_1^\mu T_{\mu\nu\lambda} = k_2^\nu T_{\mu\nu\lambda} = 0 \qquad q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu}$$

Naïve current conservation

Vector current, first diagram:

$$k_1^\mu T_{\mu\nu\lambda} > \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\not{k}_1 \frac{i}{\not{p}-m}\right]$$

use trick:

$$\not{k}_1 = (\not{p}-m) - ((\not{p}-\not{k}_1)-m)$$

we get:

$$= i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\right] - i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{\not{p}-m}\right]$$

Naïve current conservation

Vector current, first diagram:

$$k_1^\mu T_{\mu\nu\lambda} > \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\not{k}_1 \frac{i}{\not{p}-m}\right]$$

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same trick with the second diagram gives

$$= i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{\not{p}-m}\right] - i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{k}_2)-m}\gamma_\nu \frac{i}{\not{p}-m}\right]$$

Naïve current conservation

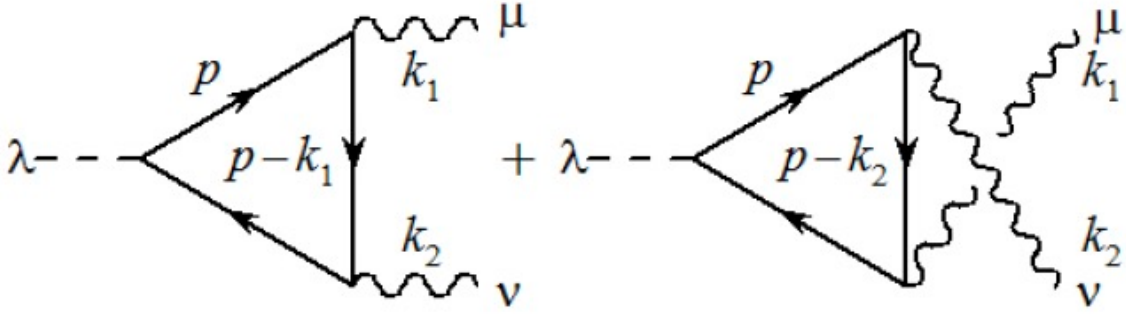
$$k_1^\mu T_{\mu\nu\lambda} \sim \int \frac{d^4p}{(2\pi)^4}$$

$$\left\{ \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not k_2) - m}\gamma_\nu \frac{i}{\not p - m}\right]\right\}$$

change variable in the first integral $p \to p + k_1$

It seems we get zero

Naïve current conservation



$$q = k_1 + k_2$$

Skipping coupling constants (charges) the amplitude reads:

$$\begin{aligned} T_{\mu\nu\lambda} \;&=\; -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma_\lambda\gamma_5\frac{i}{(\not{p}-\not{q})-m}\gamma_\nu\frac{i}{(\not{p}-\not{k}_1)-m}\gamma_\mu\right] \\ &\quad -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma_\lambda\gamma_5\frac{i}{(\not{p}-\not{q})-m}\gamma_\mu\frac{i}{(\not{p}-\not{k}_2)-m}\gamma_\nu\right] \end{aligned}$$

Naively we expect:

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu}$$

Axial current

To calculate $q^\lambda T_{\mu\nu\lambda}$

we use the following trick:

$$\begin{aligned} \not{q}\gamma_5 &= -\gamma_5\not{q} \\ &= \gamma_5\left[(\not{p}-\not{q})-m\right]-\gamma_5\left[\not{p}-m\right] \\ &= \gamma_5\left[(\not{p}-\not{q})-m\right]+\left[\not{p}-m\right]\gamma_5+2m\gamma_5 \end{aligned}$$

and for the first diagram we obtain

$$\begin{aligned} q^\lambda\left[\frac{i}{\not{p}-m}\gamma_\lambda\gamma_5\frac{i}{(\not{p}-\not{q})-m}\right] &= 2m\frac{i}{\not{p}-m}\gamma_5\frac{i}{(\not{p}-\not{q})-m} \\ &\quad +i\frac{i}{\not{p}-m}\gamma_5+i\gamma_5\frac{i}{(\not{p}-\not{q})-m} \end{aligned}$$

Axial current

Sum from the two diagrams $$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}$$

$$\begin{aligned} &\Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \\ = &\int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma_5\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\gamma_\mu + \gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\gamma_\mu\right] \\ + &\int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma_5\gamma_\mu \frac{i}{(\not{p}-\not{k}_2)-m}\gamma_\nu + \gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\mu \frac{i}{(\not{p}-\not{k}_2)-m}\gamma_\nu\right] \end{aligned}$$

Axial current

$$\Delta^{(1)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma_5\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\gamma_\mu - \frac{i}{(\not{p}-\not{k}_2)-m}\gamma_5\gamma_\nu \frac{i}{(\not{p}-\not{q})-m}\gamma_\mu\right]$$

$$\Delta^{(2)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma_5\gamma_\mu \frac{i}{(\not{p}-\not{k}_2)-m}\gamma_\nu - \frac{i}{(\not{p}-\not{k}_1)-m}\gamma_5\gamma_\mu \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu\right]$$

The question is: are $\Delta^{(1,2)}_{\mu\nu}$ equal zero?

Changing variables $p \to p + k_2$

$p \to p + k_1$

seems to nullify $\Delta^{(1,2)}_{\mu\nu}$.

However, $\Delta^{(1,2)}_{\mu\nu} \sim \int^{\infty} dp p^3 \frac{1}{p^2} \sim \int^{\infty} dp p$ are UV divergent

Due to the angular integration divergence is only linear.
What is the difference of two linearly divergent integraks?

Mathematical diggression

Consider the integral that is naively zero:

$$\int_{-\infty}^{\infty} dx\,[f(x+a) - f(x)]$$

However, if

$$f(\pm\infty) \neq 0.$$

we can calculate this integral by Taylor expansion:

$$\int_{-\infty}^{\infty} dx\,[f(x+a) - f(x)] = a\,[f(\infty) - f(-\infty)] + \frac{a^2}{2}\,[f'(\infty) - f'(-\infty)] + \ldots$$

it may happen that $\neq 0$

Mathematical diggression

Consider Euclidean integral: $$\Delta(\vec{a}) = \int d^n\vec{r}\,[f(\vec{r}+\vec{a}) - f(\vec{r})]$$

expand in a $$= \int d^n\vec{r}\,\vec{a}\cdot\vec{\nabla} f(\vec{r}) + \dots$$

apply Gauss theorem $$= \vec{a}\cdot\vec{n}\,S_n(R)\,f(\vec{R})$$

where $\vec{n} = \frac{\vec{R}}{R}$ and $S_n(R)$ is a surface of the n sphere, R is regulator.

For even n

$$S_n(R) = \frac{2\pi^{n/2}}{(n/2-1)!}R^{n-1} = \begin{cases} 2\pi R & \text{for} \quad n=2 \\ 2\pi^2 R^3 & \text{for} \quad n=4 \end{cases}$$

In Minkowski space

$$\Delta(a) = 2i\pi^2 a^\mu \lim_{R\to\infty} R^2 R_\mu\, f(R)$$