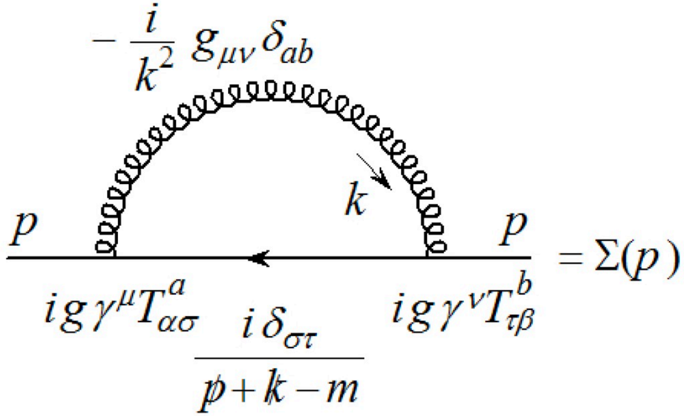
QCD lecture 3

November 3

Quark self - energy



$4 \to d = 4 - 2\varepsilon$

$$\Sigma(p) = -g^2 \mu^{4-d} C_F \delta_{\alpha\beta} \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu(\not p + \not k + m)\gamma_\mu}{[(p+k)^2 - m^2]\; k^2}$$

We want to keep the same dimensionality of Σ and *g* in any number of physical dimensions.
We therefore introduce a dimensionfull parameter μ to correct for this.

We will extend Dirac algebra by simply using $\quad g_{\mu\nu} g^{\mu\nu} = d$
It can be shown that we can treat Dirac bispinors as 4-dimensional.

Dimensional regularization preserves gauge invarince, but has problems in theories
with γ_5. This is not the case of QCD.

In the following we shall keep $m = 0$.

Integrals

$$\Sigma(p) = 2(1-\varepsilon)\, g^2 \mu^{2\varepsilon} C_F \delta_{\alpha\beta} \int \frac{d^d k}{(2\pi)^d} \frac{\not{p} + \not{k}}{(p+k)^2\, k^2}$$
$$= 2(1-\varepsilon)\, g^2 \mu^{2\varepsilon} C_F \delta_{\alpha\beta} \left[\not{p} I + \gamma_\mu I^\mu\right]$$

Define two integrals

$$\{I, I^\mu\} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p+k)^2\, k^2} \{1, k^\mu\}$$

Result:

$$\{I, I^\mu\} = i \frac{1}{2^4 \pi^2} \left(\frac{4\pi e^{-\gamma}}{-p^2}\right)^{\varepsilon} \left\{1, -\frac{1}{2} p^\mu\right\} (1+2\varepsilon) \frac{1}{\varepsilon}$$

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$$= 2(1-\varepsilon)\, g^2 \mu^{2\varepsilon} C_F \delta_{\alpha\beta} \left[\not{p} I + \gamma_\mu I^\mu\right]$$

p^2 and μ^2 combine into a dimensionless ratio

both integrals are proportional to $\not{p}$

$$\{I, I^\mu\} = i \frac{1}{2^4 \pi^2} \left(\frac{4\pi e^{-\gamma}}{-p^2}\right)^\varepsilon \left\{1, -\frac{1}{2} p^\mu\right\} (1 + 2\varepsilon) \frac{1}{\varepsilon}$$

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$$\boxed{\Sigma(p) = i\, C_F \delta_{\alpha\beta} \frac{g^2}{2^4 \pi^2} \left(\frac{\mu^2 4\pi e^{-\gamma}}{-p^2}\right)^{\varepsilon} \not{p}\, (1-\varepsilon)(1+2\varepsilon) \frac{1}{\varepsilon}}$$

$$\{I, I^\mu\} = i \frac{1}{2^4 \pi^2} \left(\frac{4\pi e^{-\gamma}}{-p^2}\right)^{\varepsilon} \left\{1, -\frac{1}{2} p^\mu\right\} \underline{(1+2\varepsilon)\frac{1}{\varepsilon}}$$

Quark self -energy

$$\Sigma(p) = i\,C_F\delta_{\alpha\beta}\frac{g^2}{2^4\pi^2}\left(\frac{\mu^2 4\pi e^{-\gamma}}{-p^2}\right)^{\varepsilon}\not{p}\,(1-\varepsilon)(1+2\varepsilon)\frac{1}{\varepsilon}$$

$$= i\not{p}\,C_F\delta_{\alpha\beta}\frac{\alpha_s}{4\pi}\left(\frac{\bar{\mu}^2}{-p^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon}+1\right)$$

we have defined:

$$\alpha_s = \frac{g^2}{4\pi} \qquad \bar{\mu}^2 = \mu^2 4\pi e^{-\gamma}$$

Quark self -energy

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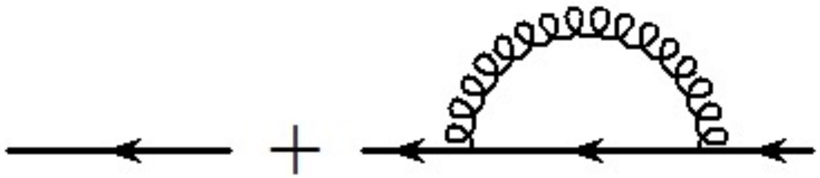
remark:

$$\left(\frac{\bar{\mu}^2}{-p^2}\right)^{\varepsilon}\frac{1}{\varepsilon} = \left(\frac{\mu^2}{-p^2}\right)^{\varepsilon} \exp\left(\varepsilon \log\left(4\pi e^{-\gamma}\right)\right)\frac{1}{\varepsilon} = \left(\frac{\mu^2}{-p^2}\right)^{\varepsilon}\left[\frac{1}{\varepsilon} + \log\left(4\pi e^{-\gamma}\right)\right]$$

this defines the MS-bar renormalization scheme

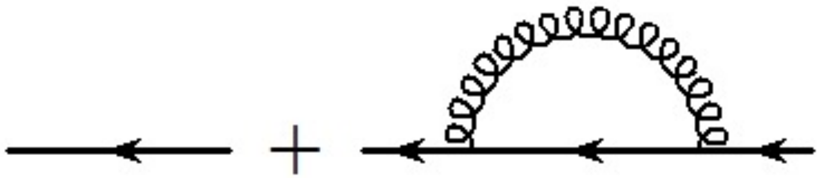
Full quark propagator



$$S_F(p) = \frac{i}{\not{p}} + \frac{i}{\not{p}}\Sigma(p)\frac{i}{\not{p}}$$

$$= \frac{i}{\not{p}}\left(1 - \frac{\alpha_s}{4\pi}C_F\left(\frac{\bar{\mu}^2}{-p^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon} + 1\right)\right)$$

Renormalization



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$$= \frac{i}{\not p}\left(1 - \frac{\alpha_s}{4\pi}C_F\left(\frac{\bar{\mu}^2}{-p^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon}+1\right)\right)$$

Practical renormalization: remove only poles ($\overline{\text{MS}}$: minimal subtraction)

$$S_F^R(p) = \frac{i}{\not p}\left(1 - \frac{\alpha_s}{4\pi}C_F\left(\frac{\bar{\mu}^2}{-p^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon}+1\right) + \frac{\alpha_s}{4\pi}C_F\frac{1}{\varepsilon}\right)$$

$$= \frac{i}{\not p}\left(1 + \frac{\alpha_s}{4\pi}C_F\left(\ln\left(\frac{-p^2}{\bar{\mu}^2}\right) - 1\right)\right)$$

Multiplicative renormalization

The same effect can be obtained by multiplivation of the “bare” propagator by the renormalization constant:

$$Z_2 S_F^R = S_F$$

where

$$Z_2 = 1 - \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2)$$

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Indeed

inverse of Z_2 $\quad \frac{1}{Z_2} \simeq 1 + \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon}$

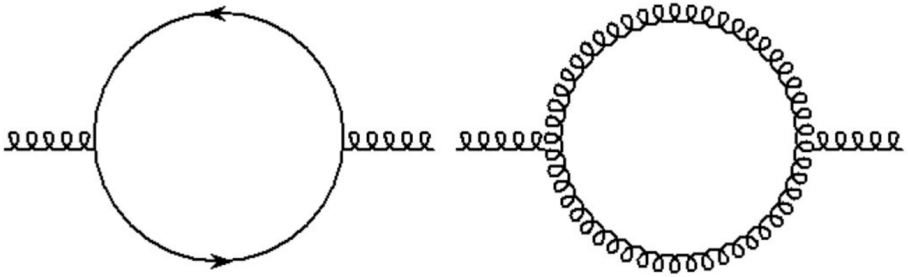
$$\begin{aligned} S_F^R &= \frac{i}{\not p}\left(1 + \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon}\right)\left(1 - \frac{\alpha_s}{4\pi} C_F \left(\frac{1}{\varepsilon} - \ln\left(\frac{-p^2}{\bar\mu^2}\right) + 1\right)\right) \\ &= \frac{i}{\not p}\left(1 + \frac{\alpha_s}{4\pi} C_F \left(\ln\left(\frac{-p^2}{\bar\mu^2}\right) - 1\right)\right). \end{aligned}$$

Multiplicative renormalization

In QFT propagator is defined as:

$$S_F(x-y) = \langle 0 \left| T(\psi(x)\bar{\psi}(y)) \right| 0 \rangle$$

Therefore multiplicative renormalization can be achieved by multipying fermion (quark) fields by $\sqrt{Z_2}$. Analogously we will renormalize gluon self-energy, and this will lead to the multiplicative renormalization of the gluon fields.



$$Z_3 = 1 - \frac{\alpha_s}{4\pi}\left(\frac{2}{3}n_f - \frac{5}{3}C_A\right)\frac{1}{\varepsilon}$$

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Write the QCD lagrangian in terms of the *bare* fields in $d = 4 - 2\varepsilon$ dims. where everything is finite and we have canonical commutation rules.

$$\mathcal{L} = \bar{\psi}_{(0)} \left(i\not{D} - m_{(0)} \right) \psi_{(0)} - \frac{1}{4} F^{a\,\mu\nu}_{(0)} F^{a}_{(0)\mu\nu}$$

Here

$$D_\mu \psi_{(0)} \quad = \quad \left(\partial_\mu + i g_{(0)} T^a A^{a(0)}_\mu \right)$$

Multiplicative renormalization

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Define renormalized fields:

$$\sqrt{Z_2}\psi = \psi_{(0)}, \; \sqrt{Z_3} A^a_\mu = A^{a(0)}_\mu, \; \text{etc.}$$

Note that when *ε* goes to zero bare fields and ren. constants are infinite.

Rewrite Lagrangian in terms of renormalized fields:

$$\begin{aligned}
\mathcal{L} &= Z_2 \bar{\psi} \left(i \not{\partial} - m_{(0)} \right) \psi - Z_2 \sqrt{Z_3} g_{(0)} \bar{\psi} T^a \not{A}^a \psi \\
&- \frac{Z_3}{4} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right)^2 + \frac{Z_3^{3/2} g_{(0)}}{2} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu \right) f^{abc} A^b_\mu A^c_\nu \\
&- \frac{Z_3^2 g^2_{(0)}}{4} \left(f^{abc} A^b_\mu A^c_\nu \right)^2 + \ldots
\end{aligned}$$

Multiplicative renormalization

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This lagrangian has wrong normalization!
We will modify it by adding counterterms

Renormalized lagrangian

We construct the renormalized lagrangian by adding counterterms

finite, renormalized 4-dim. coupling constant

$$\mathcal{L}_R = \bar{\psi} i \partial\!\!\!/ \psi + (Z_2 - 1)\bar{\psi} i \partial\!\!\!/ \psi - g\mu^{\varepsilon}\bar{\psi} T^a A\!\!\!/^a \psi - \left(Z_2\sqrt{Z_3}g_{(0)} - g\mu^{\varepsilon}\right)\bar{\psi} T^a A\!\!\!/^a \psi + \dots$$

This lagrangian is properly normalized.

Renormalized lagrangian

We construct the renormalized lagrangian by adding counterterms

$$\mathcal{L}_R = \bar{\psi} i \partial\!\!\!/ \psi + \overbrace{(Z_2 - 1)}^{\sim \alpha_s \frac{1}{\varepsilon}} \bar{\psi} i \partial\!\!\!/ \psi - g\mu^{\varepsilon} \bar{\psi} T^a A\!\!\!/^a \psi - \underbrace{\left(Z_2 \sqrt{Z_3} g_{(0)} - g\mu^{\varepsilon} \right)}_{\sim \alpha_s \frac{1}{\varepsilon}} \bar{\psi} T^a A\!\!\!/^a \psi + \ldots$$

This lagrangian is properly normalized.

Counterterms remove singularities in loops, which allows to remove regularization. If we need only a finite number of counterterms to remove singularities in $1/\varepsilon$ to all orders of perturbation theory, then theory is renormalizable. Gauge theories are renormalizable.

Renormalized coupling constant

$$\underbrace{\left(Z_2\sqrt{Z_3}g_{(0)} - g\mu^{\varepsilon}\right)}_{\sim \alpha_s \frac{1}{\varepsilon}}$$

This equation has solution of the form

$$g_{(0)} = g\mu^{\varepsilon}\left(1 + g^2\frac{\tilde{\beta}}{\varepsilon} + \ldots\right)$$

because in the lowest order bare and renormalized g should be the same in 4 dims.

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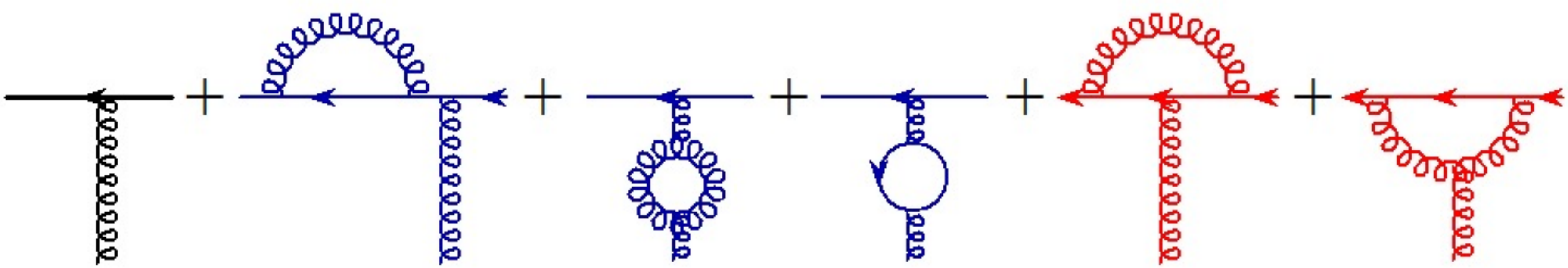

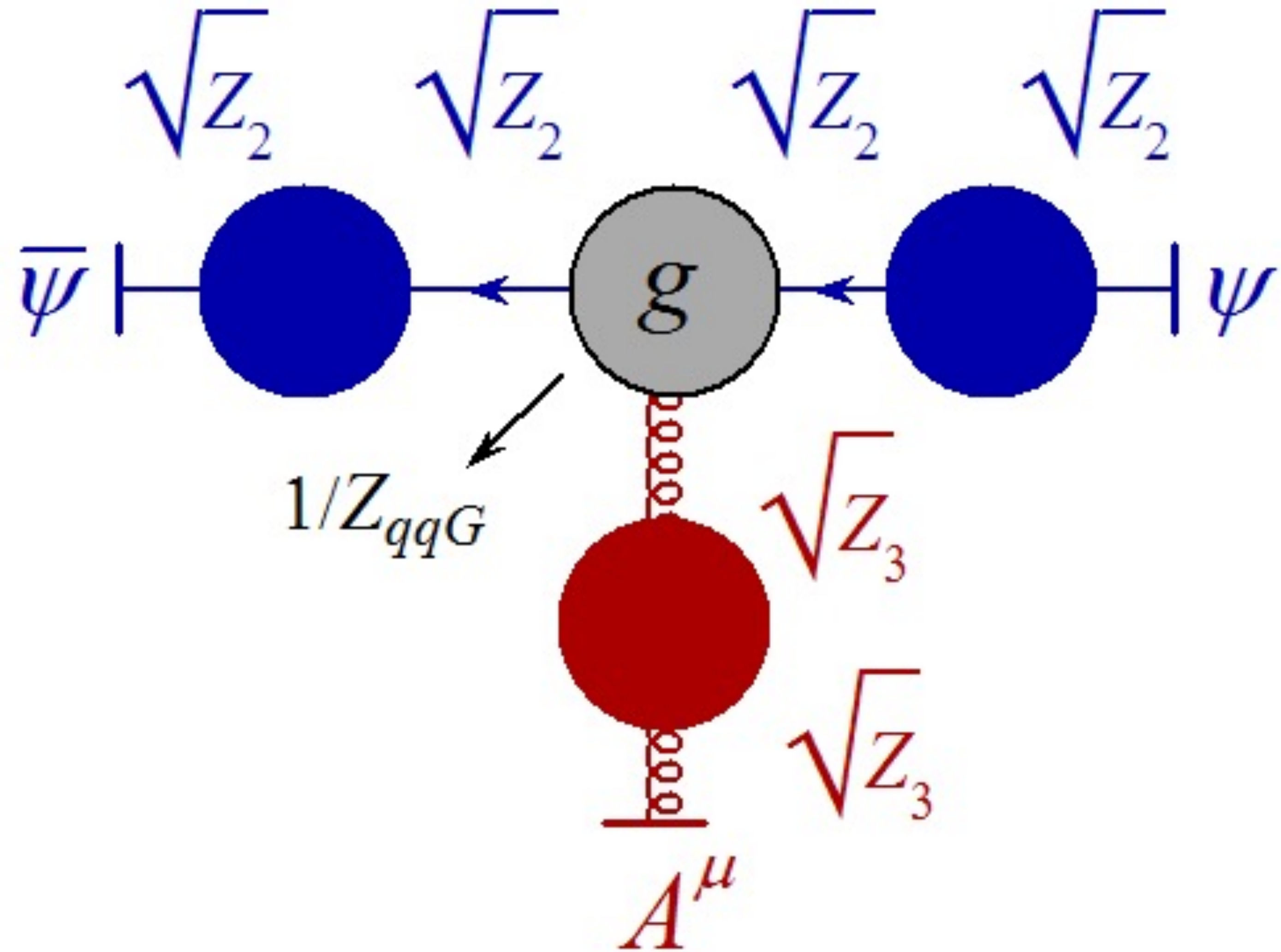

$\sqrt{Z_2}$
$\sqrt{Z_2}$
$\sqrt{Z_2}$
$\sqrt{Z_2}$
$\bar{\psi}$
g
ψ
$1/Z_{qqG}$
$\sqrt{Z_3}$
$\sqrt{Z_3}$
A^μ

Renormalized coupling constant

$$\underbrace{\left(\frac{Z_2\sqrt{Z_3}}{Z_{Gqq}}g_{(0)} - g\mu^{\varepsilon}\right)}_{\sim\, \alpha_s \frac{1}{\varepsilon}}$$

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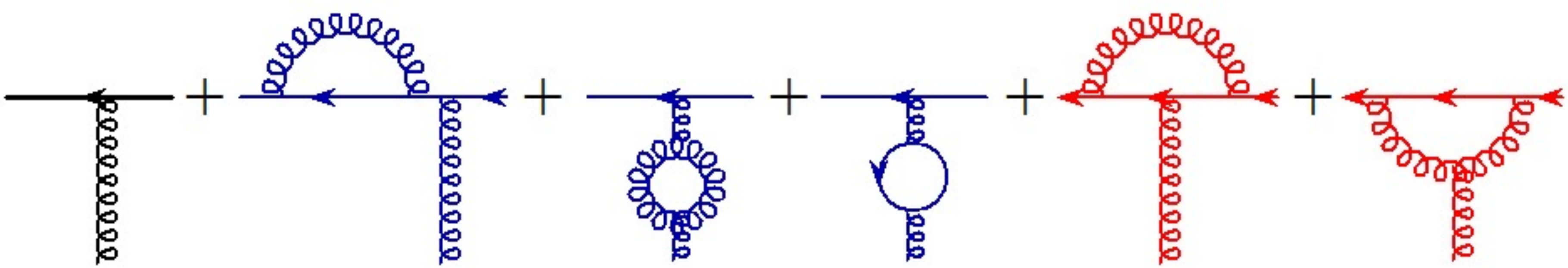
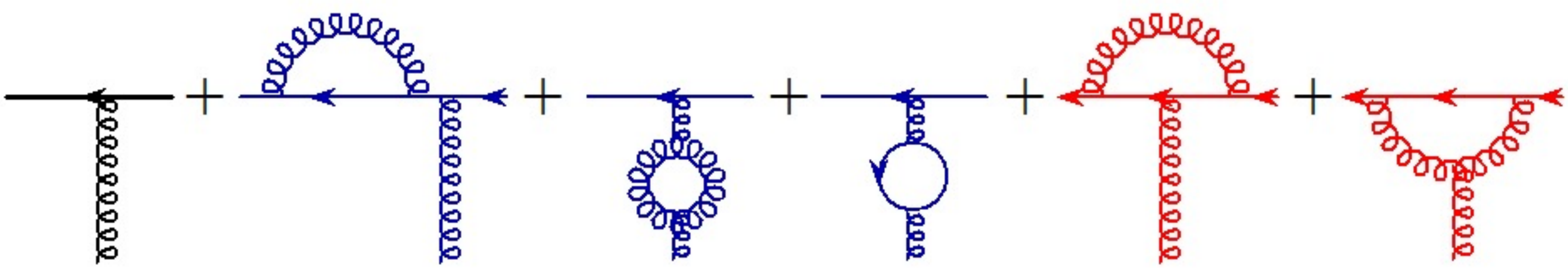


Renormalized coupling constant

Full result:

$$g_{(0)} = g\mu^{\varepsilon}\left(1 - \frac{\alpha_s}{4\pi}\left(\frac{11}{6}C_A - \frac{1}{3}n_f\right)\frac{1}{\varepsilon} + \ldots\right)$$

At this order this is gauge invariant.



Running coupling constant

$$g_{(0)} = g\mu^{\varepsilon}\left(1 - \frac{\alpha_s}{4\pi}\left(\frac{11}{6}C_A - \frac{1}{3}n_f\right)\frac{1}{\varepsilon} + \ldots\right)$$

Bare coupling constant should not depend on μ. This can be achieved only if:

$$g = g(\mu)$$

Hence we have the following equation:

$$0 = \frac{d}{d\ln\mu^2} g_{(0)}(\mu, g(\mu), \varepsilon) = \frac{\partial g_{(0)}}{\partial \ln\mu^2} + \frac{\partial g_{(0)}}{\partial g}\frac{dg(\mu)}{d\ln\mu^2}$$

$$\frac{dg(\mu)}{d\ln\mu^2} = -\frac{\partial g_{(0)}/\partial\ln\mu^2}{\partial g_{(0)}/\partial g}$$

Running coupling constant

$$g_{(0)} = g\mu^{\varepsilon}\left(1 - \frac{\alpha_s}{4\pi}\left(\frac{11}{6}C_A - \frac{1}{3}n_f\right)\frac{1}{\varepsilon} + \dots\right)$$

First calculate $\dfrac{\partial g_{(0)}}{\partial \ln \mu^2}$

We have $\mu^{\varepsilon} = \exp\left(\frac{1}{2}\varepsilon \ln \mu^2\right)$ and $\dfrac{d}{d \ln \mu^2}\mu^{\varepsilon} = \frac{1}{2}\varepsilon\mu^{\varepsilon}$

Running coupling constant

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Running coupling constant

$$g_{(0)} = g\mu^{\varepsilon}\left(1 - \frac{\alpha_s}{4\pi}\left(\frac{11}{6}C_A - \frac{1}{3}n_f\right)\frac{1}{\varepsilon} + \ldots\right)$$

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$$\frac{dg(\mu)}{d\ln \mu^2} = -\frac{1}{2}\varepsilon\frac{g_{(0)}}{\partial g_{(0)}/\partial g}$$

We need a pole part only

Running coupling constant

We usually work with $$a_s(\mu) = \frac{g^2(\mu)}{16\pi^2} = \frac{\alpha_s(\mu)}{4\pi}$$

which gives

$$\frac{da_s(\mu)}{d\ln\mu^2} = \frac{2g(\mu)}{16\pi^2}\frac{dg(\mu)}{d\ln\mu^2} = -\frac{g(\mu)}{16\pi^2}\varepsilon\frac{g_{(0)}}{\partial g_{(0)}/\partial g}\Big|_{\varepsilon=0} = \beta(a_s)$$

beta function

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$$= -\frac{g}{16\pi^2}\varepsilon\frac{g\mu^\varepsilon\left(1-\frac{\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}\right)}{\mu^\varepsilon\left(1-\frac{3\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}\right)}$$

remember
$g\,\alpha_s \sim g^3$

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$$= -a_s\varepsilon\frac{\left(1-\frac{3\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}\right)+\frac{2\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}}{1-\frac{3\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}}$$

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$$= -\frac{g}{16\pi^2}\varepsilon\frac{g\mu^\varepsilon\left(1-\frac{\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}\right)}{\mu^\varepsilon\left(1-\frac{3\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}\right)}$$

$$= -a_s\varepsilon\frac{\left(1-\frac{3\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}\right)+\frac{2\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}}{1-\frac{3\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}}$$

$$= -a_s\varepsilon - a_s^2\left(\frac{11}{3}C_A-\frac{2}{3}n_f\right)+\ldots$$

perturbative expansion in α_s

QCD beta function

$$\beta(a_s) = -a_s^2\left(\frac{11}{3}C_A - \frac{2}{3}n_f\right) + \ldots$$

Renormalization group equation (RGE):

$$\frac{da_s}{d\ln\mu^2} = \beta(a_s)$$

Generally

$$\beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 + \ldots$$

Solving RGE

$$\frac{da_s}{d\ln\mu^2} = \beta(a_s)$$

$$\ln\frac{\mu^2}{\mu_0^2} = \int\limits_{a_s(\mu_0)}^{a_s(\mu)} \frac{da_s}{\beta(a_s)}$$

Solving RGE one loop approximation

$$\frac{da_s}{d\ln\mu^2} = \beta(a_s) = -\beta_0 a_s^2$$

$$\ln\frac{\mu^2}{\mu_0^2} = \int_{a_s(\mu_0)}^{a_s(\mu)} \frac{da_s}{\beta(a_s)} = \frac{1}{\beta_0}\left(\frac{1}{a_s(\mu)} - \frac{1}{a_s(\mu_0)}\right)$$

Solving RGE one loop approximation

$$\frac{da_s}{d\ln\mu^2} = \beta(a_s) = -\beta_0 a_s^2$$

$$\ln\frac{\mu^2}{\mu_0^2} = \int_{a_s(\mu_0)}^{a_s(\mu)} \frac{da_s}{\beta(a_s)} = \frac{1}{\beta_0}\left(\frac{1}{a_s(\mu)} - \frac{1}{a_s(\mu_0)}\right)$$

Rewrite last equation in the following form:

$$\frac{1}{a_s(\mu)} - \beta_0 \ln\mu^2 = \frac{1}{a_s(\mu_0)} - \beta_0\ln\mu_0^2 = -\beta_0 \ln\Lambda_{\mathrm{QCD}}^2$$

This equation says that both sides are constant as functions of μ or μ_0.
This constant is encoded in Λ_{QCD}, which has to be taken from experiment.

Running coupling constant

We can either write the asymptotic solution

$$a_s(\mu) = \frac{1}{\beta_0 \ln \frac{\mu^2}{\Lambda^2_{\text{QCD}}}}$$

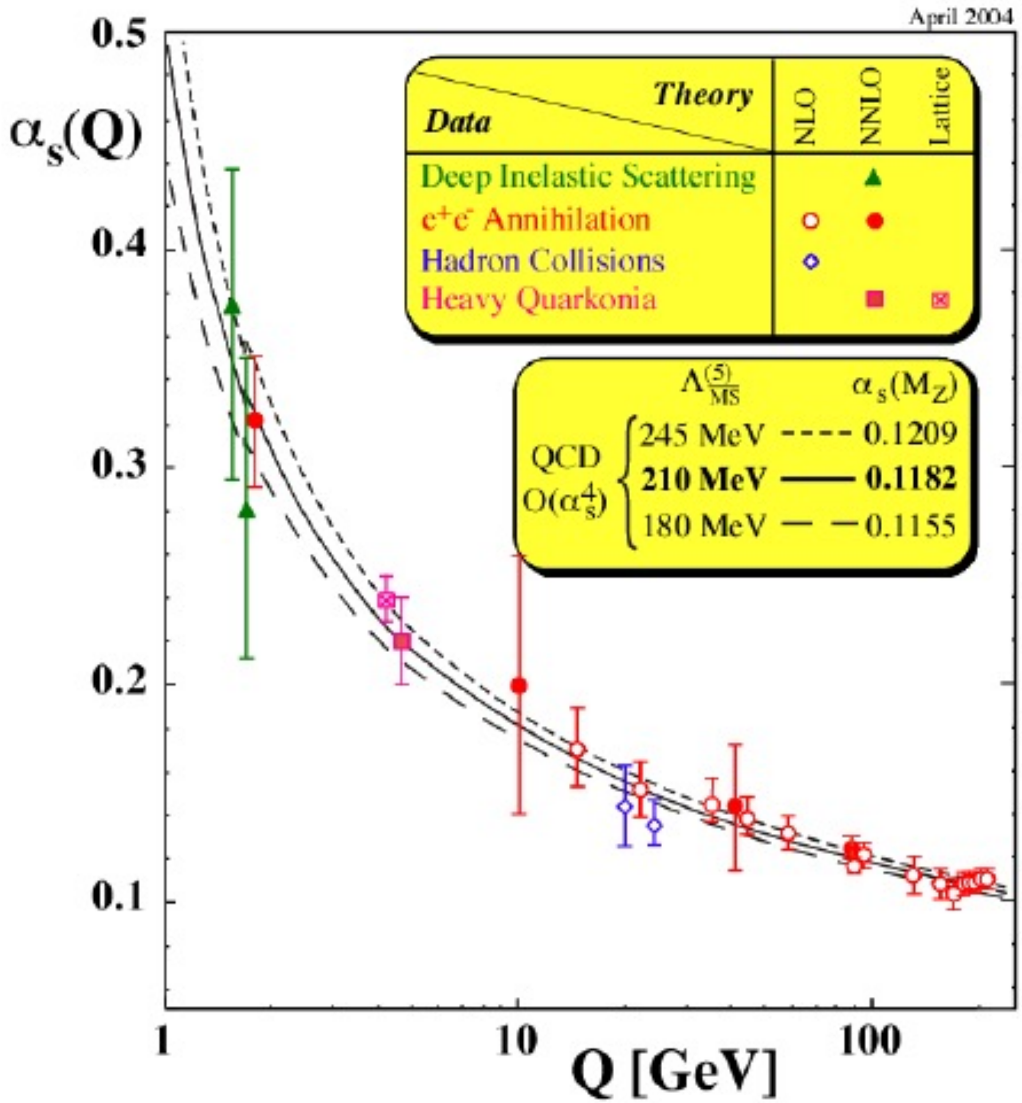
or an equation relating the couplings at two different scales:

$$a_s(\mu) = \frac{a_s(\mu_0)}{1 + \beta_0 a_s(\mu_0) \ln(\mu^2/\mu_0^2)}$$

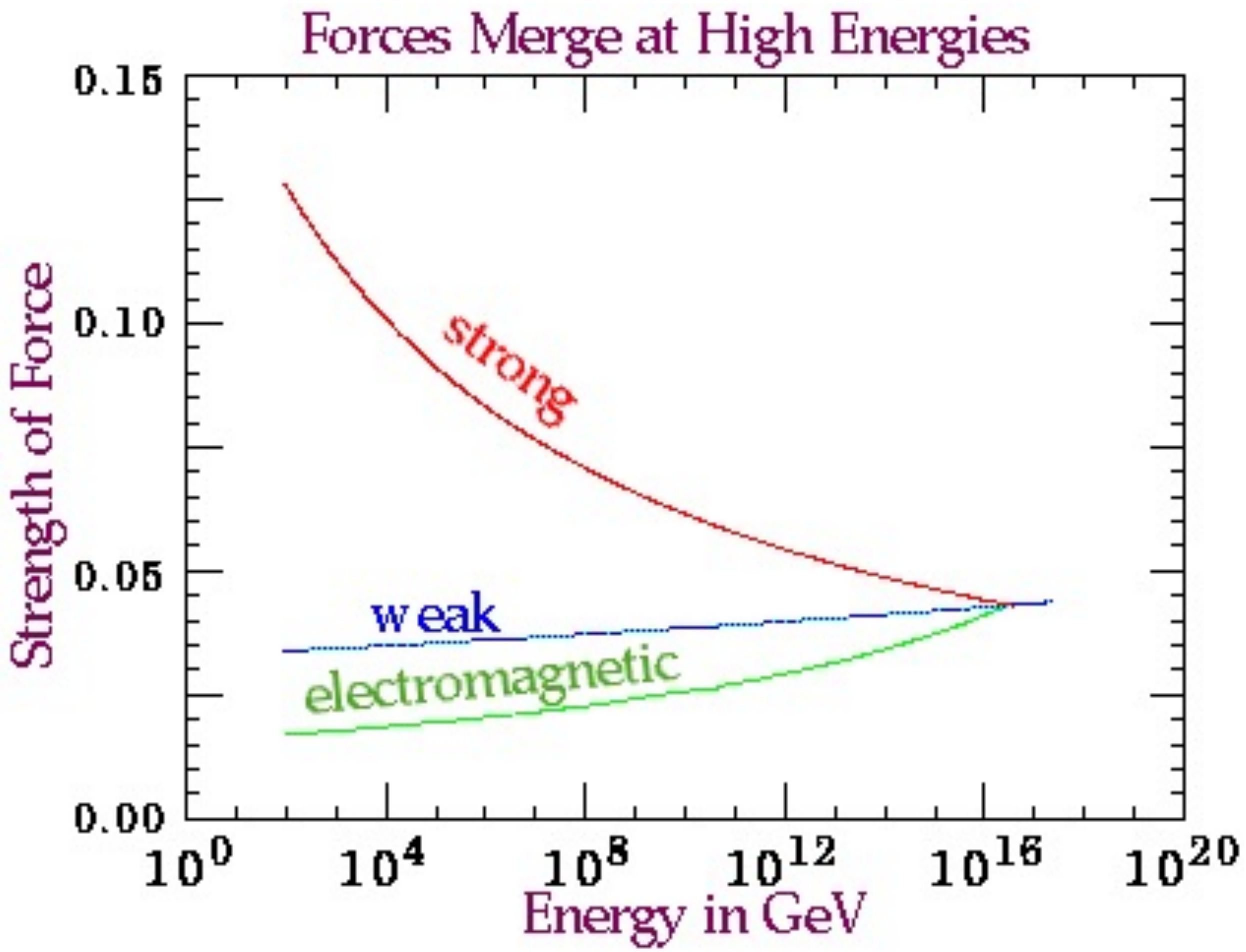
For negative β_0 there is a problem (in QED: Landau pole).

In QCD $a_s(\mu \to \infty) \to 0$ This is called asymptotic freedom.
This is why we can apply pert. theory even though the coupling is not a'prori small. This also explains why the parton model works.

Running coupling constant



Grand Uninified Theory ? (GUT)



Consequences of running

In a typical QCD calculation we can choose μ^2 at will, and a typical choice is that μ^2 corresponds to the large momentum transfer present in a given process. See for example the quark propagator (although it is not an observable):

$$S_F^R = \frac{i}{\not p}\left(1 + \frac{\alpha(\mu^2)}{4\pi} C_F \left(\ln\left(\frac{-p^2}{\bar{\mu}^2}\right) - 1\right)\right)$$

Choice:

$$\bar{\mu}^2 \sim -p^2 \;(\text{provided } -p^2 \gg \Lambda^2_{\text{QCD}})$$

nullifies large logarithm.

One might be worried that the change of scale changes the numerical value of the quark propagator in plain contradiction with the RG invariance. One should, however, keep in mind that RG invariance concerns *full theory*, and here we are dealing with one loop approximation only. In two, three *etc*. loop calculations sensitivity to the choice of scale is significantly reduced.

Renormalization: summary

- Ultraviolet infinities appear in loop diagrams
- Regularization, usefull method: dimensional regularization
- Renormalization constants: fields, couplings, masses
- Relations between renormalization constants

$$\frac{Z_2\sqrt{Z_3}}{Z_{Gqq}} = \frac{Z_3^{3/2}}{Z_{GGG}}$$

- Counterterms (finite # - theory is renormalizable)
- Dimensional transmutation: Λ_{QCD}
- Running couplings and masses
- Asymptotic freedom
- Scale choice may minimize h.o. correctons
- Only full theory is scale invariant