QCD Lecture 4

October 26

Quantum Chromo Dynamics

$$\Psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix}$$

Gauge theory based on SU(3) group

$$\Psi(x) \to \Psi'(x) = U(x)\Psi(x) \qquad U(x) = e^{-i\theta_m(x)T^m} \qquad (m = 1, 2, \ldots N^2 - 1)$$

covariant derivative

$$D_\mu = \partial_\mu + igT^m A_\mu^m(x) = \partial_\mu + ig\boldsymbol{A}_\mu(x)$$

transforms as

$$D'_\mu = U(x) D_\mu U^\dagger(x) \longrightarrow$$

$$\longrightarrow \boldsymbol{A}'_\mu(x) = U(x)\boldsymbol{A}_\mu(x)U^\dagger(x) - \frac{i}{g}U(x)\partial_\mu U^\dagger(x)$$

SU(N) group

in fundamental representation generators are given as N x N hermitean matrices that satisfy commutation relations

$$[T_m, T_n] = i f_{mnl} T_l$$

f_{mnl} are totally antisymmetric tensors known as structure constants. To define the group we either give explicit form of the generators or a complete set of structure constants.

Examples:
SU(2)
Pauli matrices

$$T^i = \frac{1}{2}\tau^i$$

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Normalization:

$$\mathrm{Tr}(T_m T_n) = \frac{1}{2}\delta_{mn}$$

SU(N) group

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Examples:
SU(3)
Gell-Mann
matrices

$$T^i = \frac{1}{2}\lambda^m$$

$$\lambda^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda^2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda^5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix},$$

$$\lambda^6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \lambda^7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix},$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Conjugated fundamental rep.

obviously, there are infintely many matrix representations related by the unitary transformation

$$T_n' = U^\dagger T_n U$$

let's complex conjugate the commutation relation

$$[T_m, T_n] = i f_{mnl} T_l$$

and multiply all generators by minus

$$[-T_m^*, -T_n^*] = i f_{mnl} \left(-T_l^*\right)$$

we have constructed conjugated representation $T_n' = -T_n^*$ satysfying commutation relation

is this representation unitary equivalent to the fundamental one?

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is this representation unitary equivalent to the fundamental one?

SU(2) – yes
SU(3) and higher – no

complication

$$\tau_i \tau_j = \delta_{ij} + i\varepsilon_{ijk}\tau_k,$$

$$\lambda_a \lambda_b = \frac{2}{3}\delta_{ab} + i f_{abc}\lambda_c + d_{abc}\lambda_c$$

therefore quarks and antiquarks are different objects

Adjoint representation

it follows from the Jacobi identity

$$[T_m, [T_n, T_l]] + [T_n, [T_l, T_m]] + [T_l, [T_m, T_n]] = 0$$

that

$$f_{nlk} f_{kmr} + f_{lmk} f_{knr} + f_{mnk} f_{klr} = 0$$

this relation can be writen in terms of $(N^2\text{-}1) \times (N^2\text{-}1)$ matrices defined as

$$\left(T_l^{\text{adj}}\right)_{mn} = -i f_{lmn}$$

in the following way

$$[T_m, T_n] = i f_{mnl} T_l$$

which means that T_l^{adj} are SU(3) generators, they form adjoint representation
note that

$$-T_l^{\text{adj}\,*} = T_l^{\text{adj}}$$

so adjoint representation is self-conjugated (real)

Adjoint representation

consider vector in the adjoint representation $A = (a^1, \dots, a^{N^2-1})$

which transforms as $A' = U^{\mathrm{adj}} A \to a'^{\,m} = a^m - \theta^l f_{lmn} a^n + \dots$.

because $U(x) = e^{-i\theta_m(x) T^m}$ and $\left(T_l^{\mathrm{adj}}\right)_{mn} = -i f_{lmn}$

one can write this transformation differently, defining $\boldsymbol{A} = \sum_{n=1}^{N^2-1} a^n T_n$

then $\boldsymbol{A}' = U \boldsymbol{A} U^\dagger$

leads to

$$\begin{aligned} a'^{\,m} T_m &= (1 - i\,\theta^n T_n + \dots) a^m T_m (1 + i\,\theta^n T_n + \dots) \\ &= a^m T_m - i\,\theta^n \,[T_n, T_m]\, a^m \\ &= a^m T_m + \theta^n\, f_{nmk} T_k\, a^m \\ &= \left(a^m - \theta^l\, f_{lmn}\, a^n\right)\, T_m , \end{aligned}$$

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gauge fields transform according to the adjoint representation of SU(N)

QED vs. QCD

field tensor in QED $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

can be expressed in terms of covariant derivatives, because the the field is Abelian:

$$F^{\mu\nu} = D^\mu A^\nu - D^\nu A^\mu = (\partial^\mu + iq\underline{A^\mu)\,A^\nu} - (\partial^\nu + iq\underline{A^\nu)\,A^\mu}$$

this can be generalized to the non Abelian case where the commutator does not vanish

$$\boldsymbol{F}_{\mu\nu} = D_\mu \boldsymbol{A}_\nu - D_\nu \boldsymbol{A}_\mu = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + ig\left[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu\right]$$

in order to find transformaion law, we have first to prove that

$$\boldsymbol{F}_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu]$$

commutator is in principle an operator
and the field tensor is a function!

because

$$D'_\mu = U(x) D_\mu U^\dagger(x)$$

we have

$$\boldsymbol{F}'_{\mu\nu} = U(x)\boldsymbol{F}_{\mu\nu}U^\dagger(x)$$

QCD Lagrangian

gauge boson part (yang-Mills)

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\,\mathrm{Tr}(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}) = -\frac{1}{4}\sum_m F^m_{\mu\nu}F^{m\,\mu\nu}$$

in QED $(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$

in QCD $(\partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + ig[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu])^2$

QCD lagrangian contains interactions!
gluons interact with themselves, they carry
adjoint color charge

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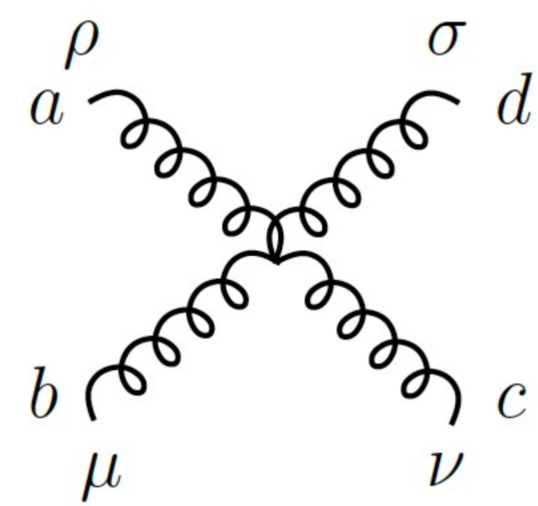

QCD lagrangian contains interactions!
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$$-ig_s^2 f^{abe} f^{cde}\,(g_{\rho\nu}g_{\mu\sigma} - g_{\rho\sigma}g_{\mu\nu})$$
$$-ig_s^2 f^{ace} f^{bde}\,(g_{\rho\mu}g_{\nu\sigma} - g_{\rho\sigma}g_{\mu\nu})$$
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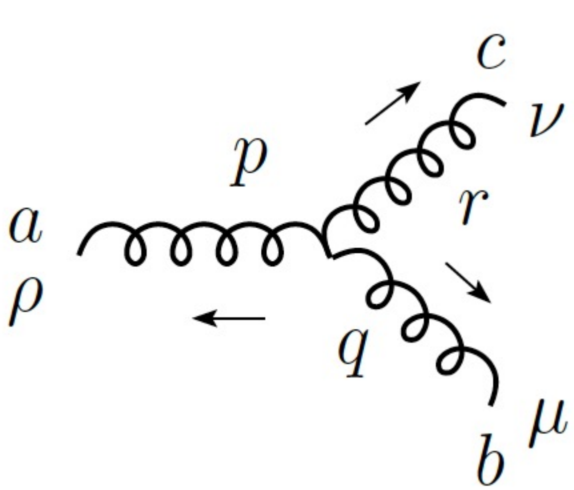

$$-g_s f^{abc}\,[(p-q)_\nu\, g_{\rho\mu} + (q-r)_\rho\, g_{\mu\nu} + (r-p)_\mu\, g_{\nu\rho}]$$

QCD Lagrangian

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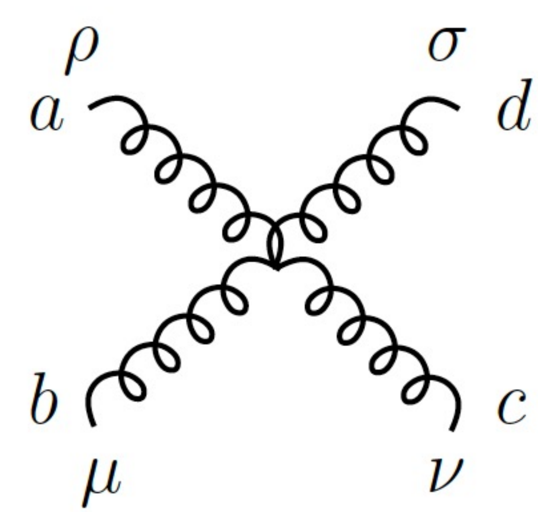

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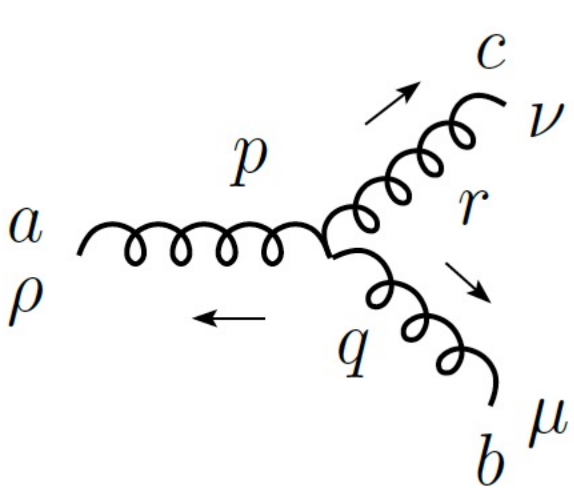

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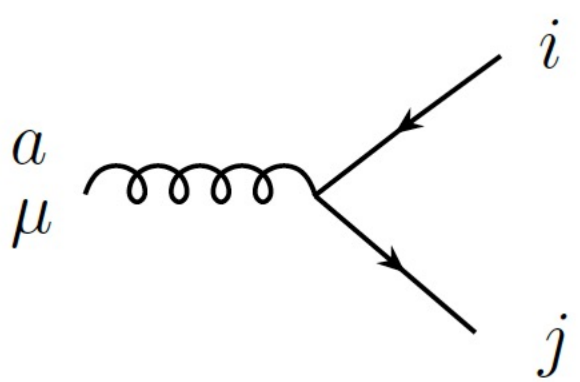
Full QCD lagrangian

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left[\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right] + \sum_{f=1}^{6}\left[\overline{q}_f\, i\gamma^{\mu} D_{\mu} q_f - m_f \overline{q}_f\, q_f\right]$$

quarks interact via covariant derivative



$$i g_s \gamma_\mu T^a_{ji}$$

propagators:

$$i S_F(p) = \cdot\; i\,\delta_{ij}\frac{(\not{k} + m)}{k^2 - m^2 + i\epsilon}$$

gauge choice!

$$i D_F(p)_{\mu\nu} = \frac{-i\,\delta_{ab}}{k^2 + i\epsilon}\left[g_{\mu\nu} - (1-\eta)\frac{k_\mu k_\nu}{k^2}\right]$$

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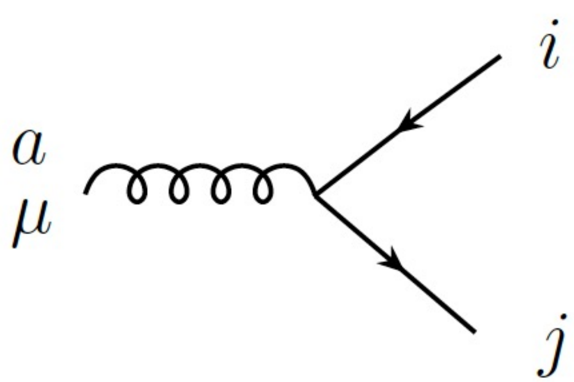

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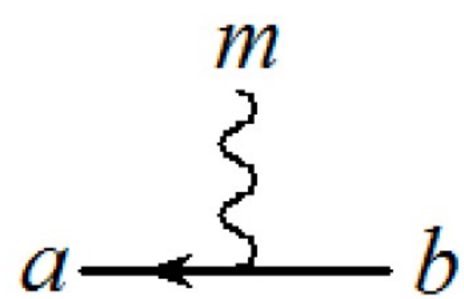
Color factors

each Feynman diagram is a product of a momentum-Dirac structure (like in QED) and a color factor

to calculate color factors it is very practical to use the graphical notation

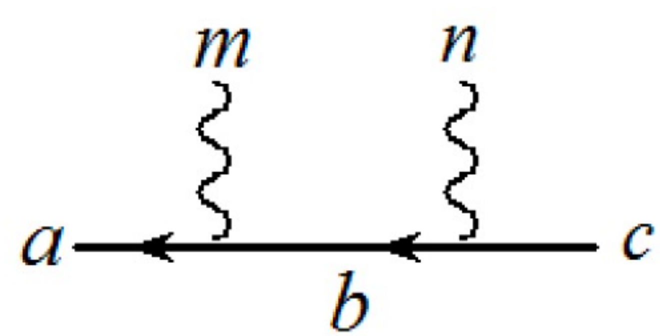
fundamental geneator:

$m,n = 1,2,\ldots N^2\text{-}1, \quad a,b = 1,2,\ldots N$



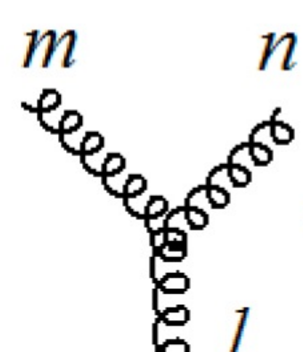
$= T^m_{ab}$

multiplication:



$= T^m_{ab}\, T^n_{bc}$

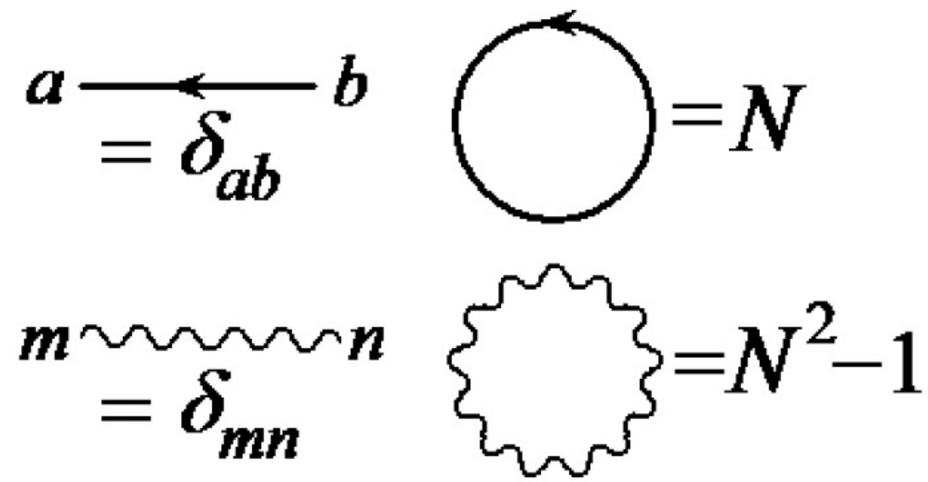
adjoint generator:



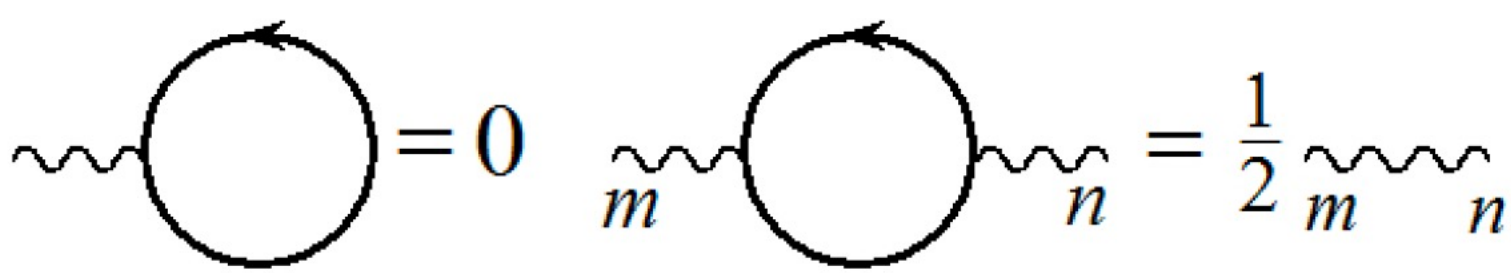
$= if_{mnl} = -if_{nml}$

Color factors

Kroneker deltas and traces:



generators are tracless and normalized to 1/2

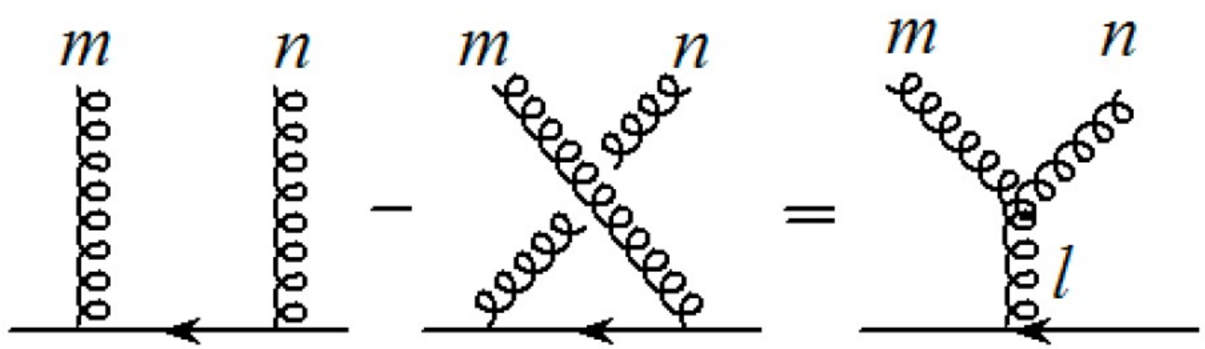


$$\mathrm{Tr}(T_m T_n) = \frac{1}{2}\delta_{mn}$$

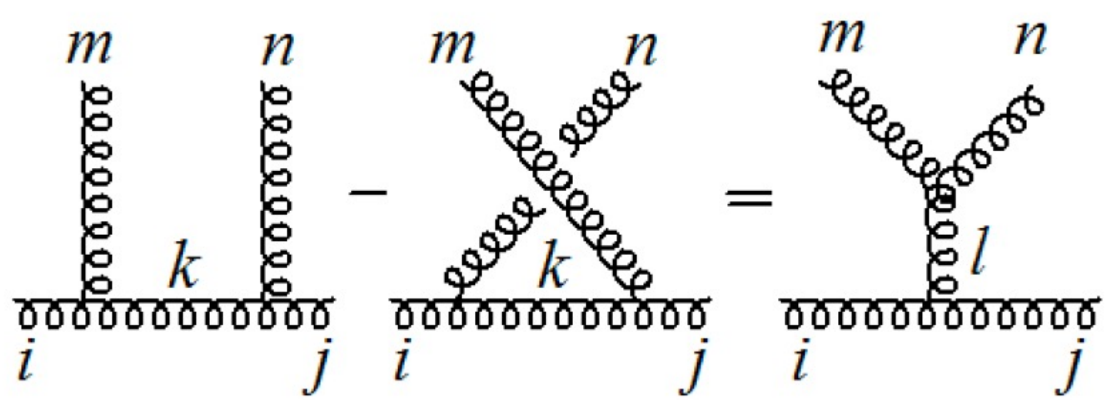
Color factors

commutation relations: $[T_m, T_n] = i f_{mnl} T_l$

fundamental:



adjoint:



Color factors

Example:

Casimir operator for the fundamental representation

quadratic Casimir operator is the sum over all generators squared
and it is proportional to unity multiplied by a number,
which is simply called "Casimir"

$$\sum_n (T^n)^2 = C_F\,\mathbf{1}$$

In SU(2) for any representation of spin *s* it is equal to

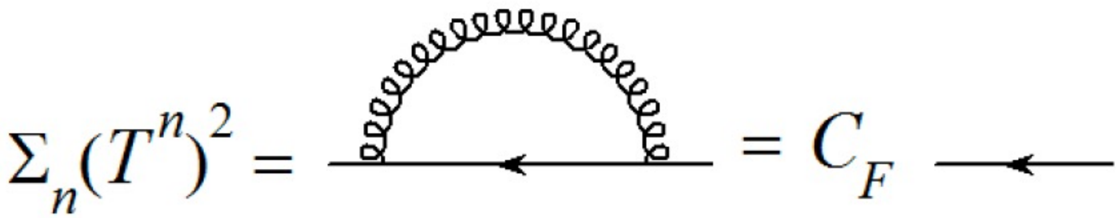
$$\sum_n \hat{S}_n^2 = s(s+1)\,\mathbf{1}$$

Color factors

Example:

Casimir operator for the fundamental representation

$$\sum_n (T^n)^2 = C_F \, \mathbf{1}$$



Color factors

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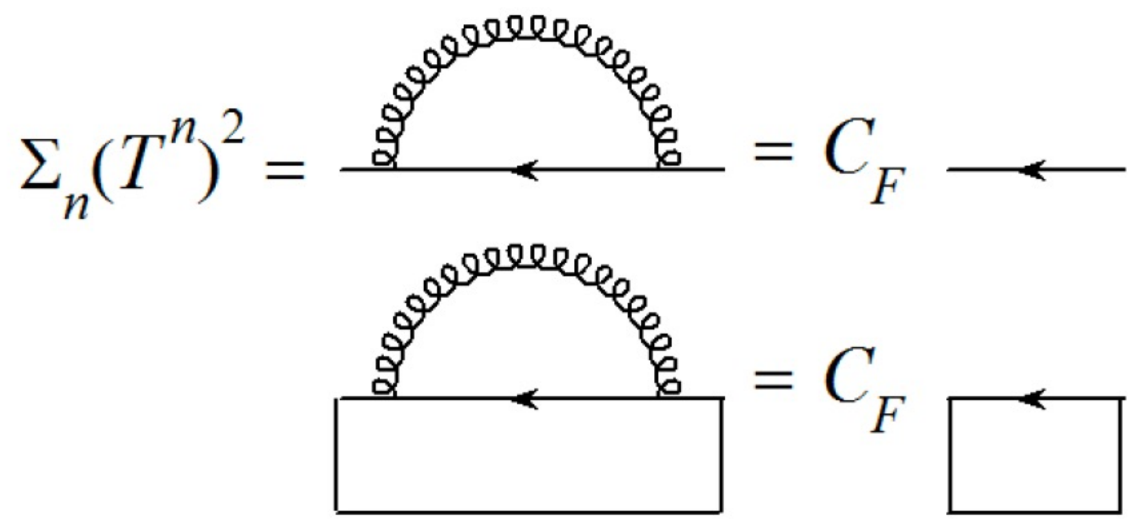
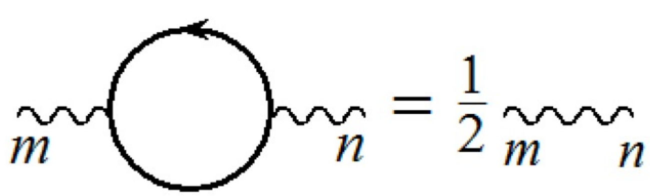
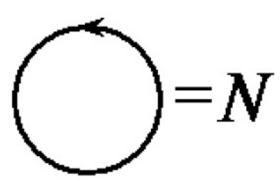
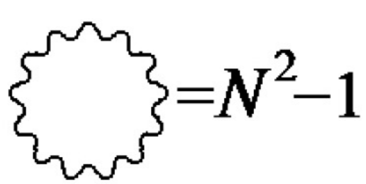


contract fermion line:

use:







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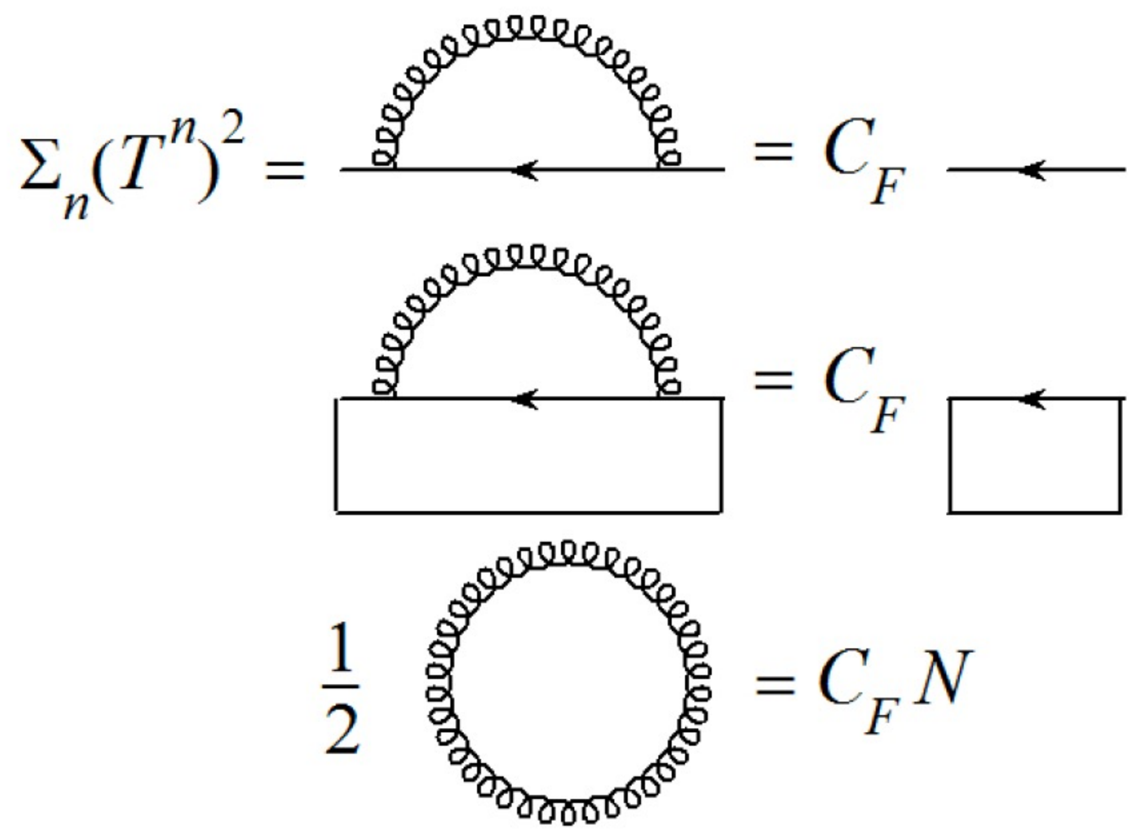

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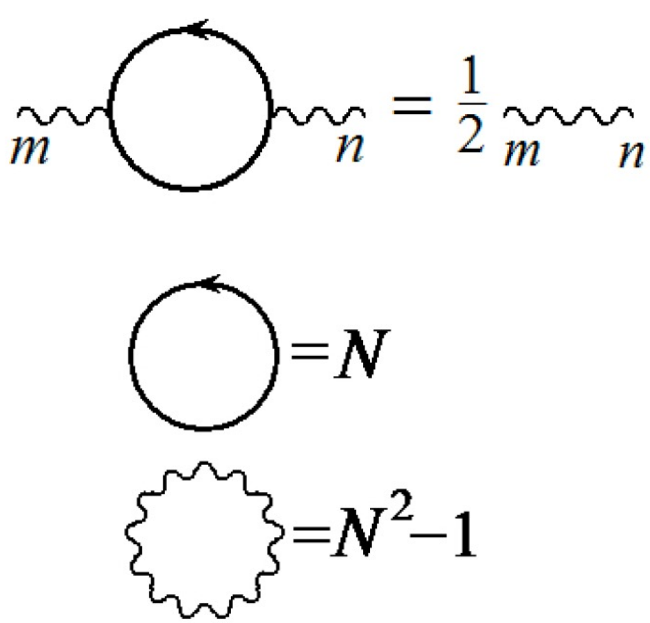

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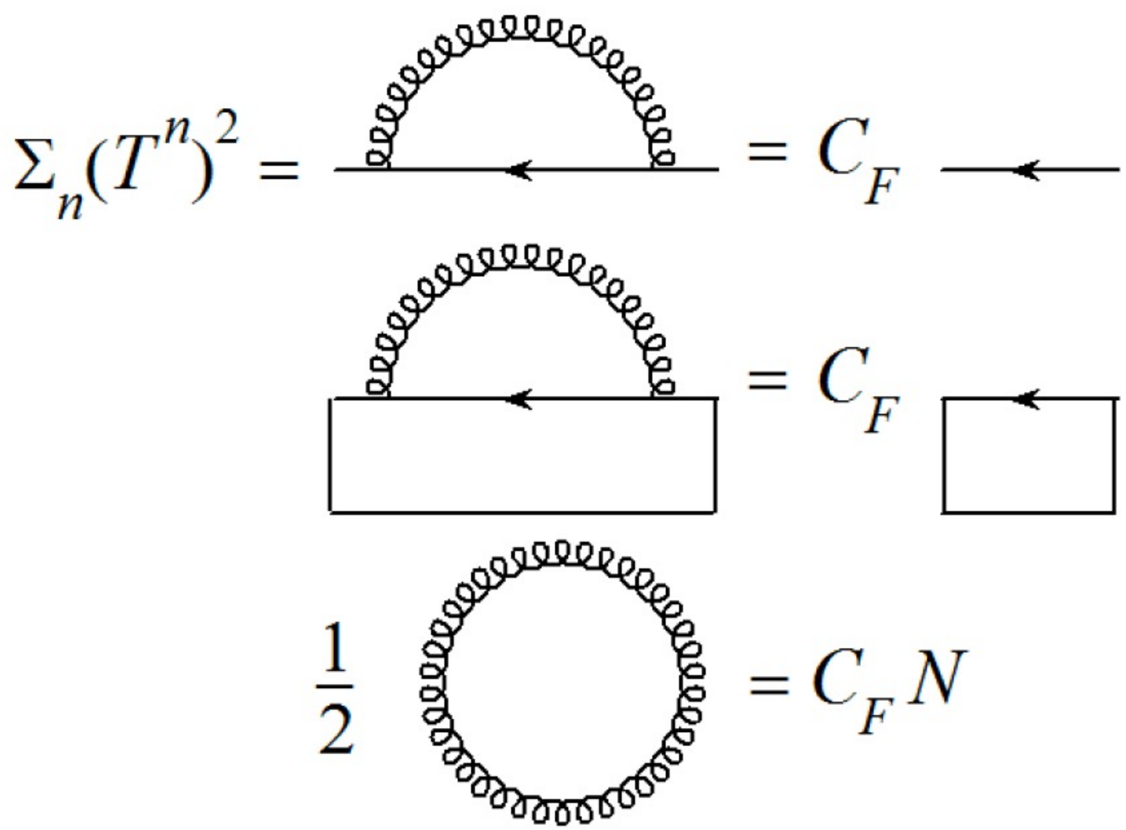

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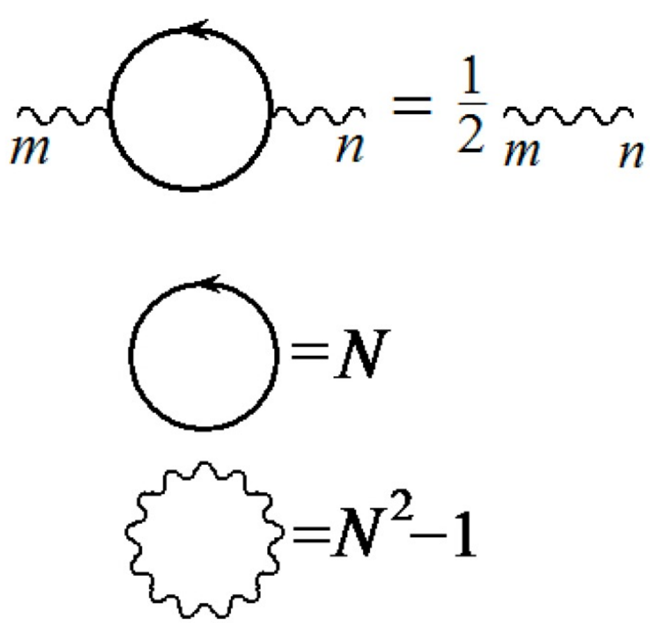

$$C_F = \frac{N^2 - 1}{2N} = \begin{cases} \frac{3}{4} & \text{SU(2)} \\ \frac{4}{3} & \text{SU(3)} \end{cases}$$