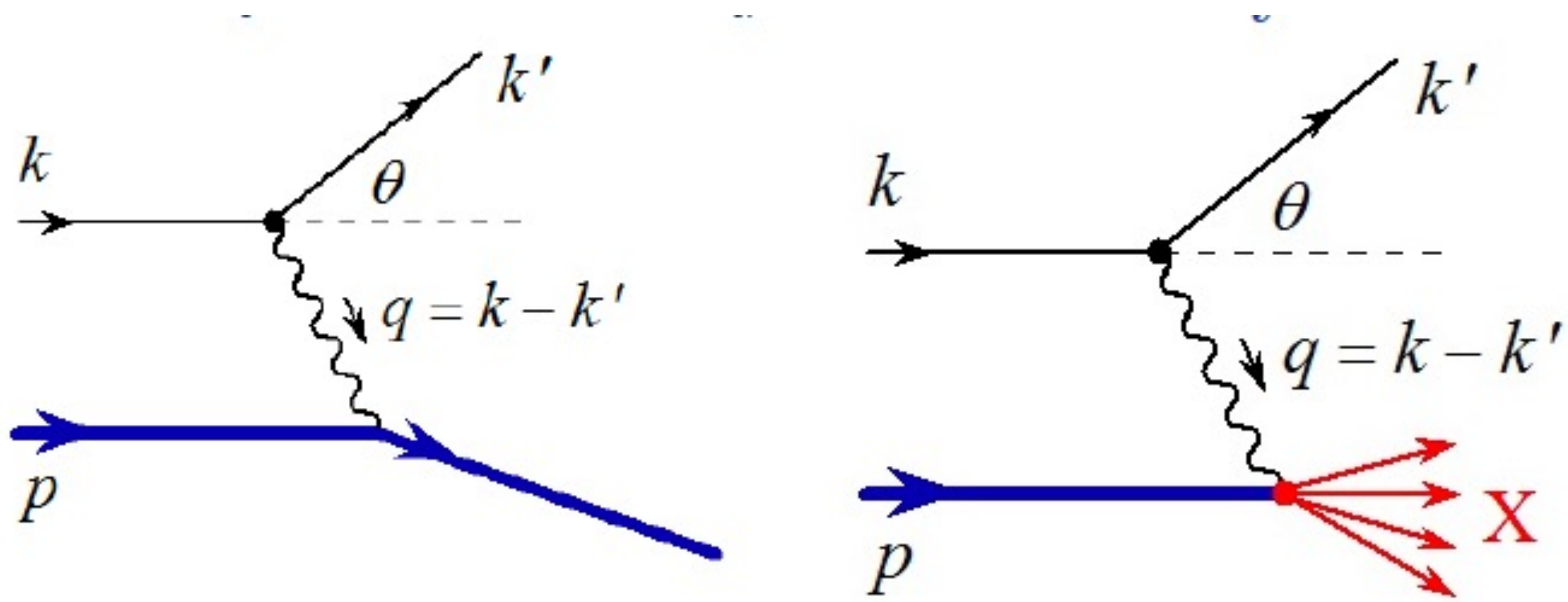
QCD Lecture 3

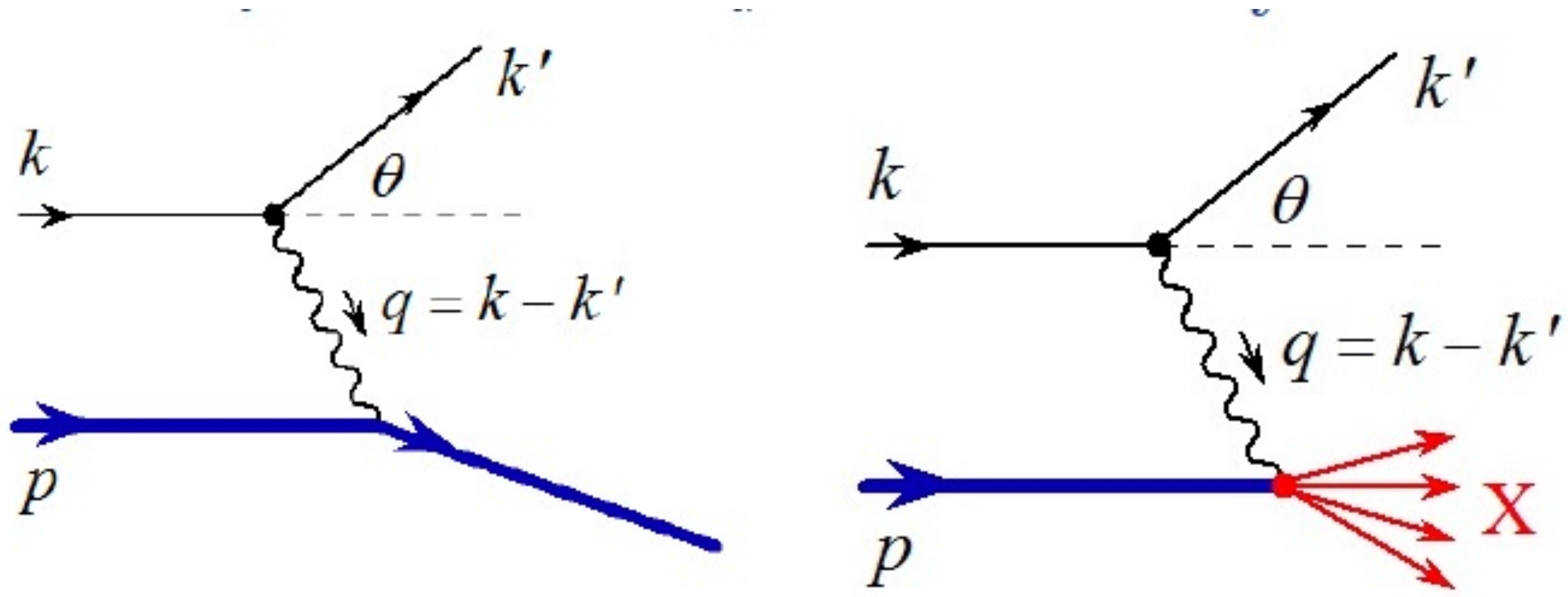
October 19

Deep Inelastic Scattering (DIS)



$$p = M(1, 0, 0, 0),$$
$$k = \omega(1, 0, 0, 1),$$
$$k' = \omega'(1,\ \sin\theta\sin\varphi,\ \sin\theta\cos\varphi,\ \cos\theta)$$
$$q = k - k' = p' - p.$$

Deep Inelastic Scattering (DIS)



4-momentum transfer and energy transfer

$$q^2 = -2\omega\omega'(1-\cos\theta) = -4\omega\omega'\sin^2\frac{\theta}{2}, \qquad \nu = \omega - \omega'$$

on mass-shell condition for scattered proton (not present in the inelastic case):

$$\delta((p+q)^2 - M^2) = \delta(2M\nu - Q^2) = \frac{1}{2M}\delta\left(\nu - \frac{Q^2}{2M}\right)$$

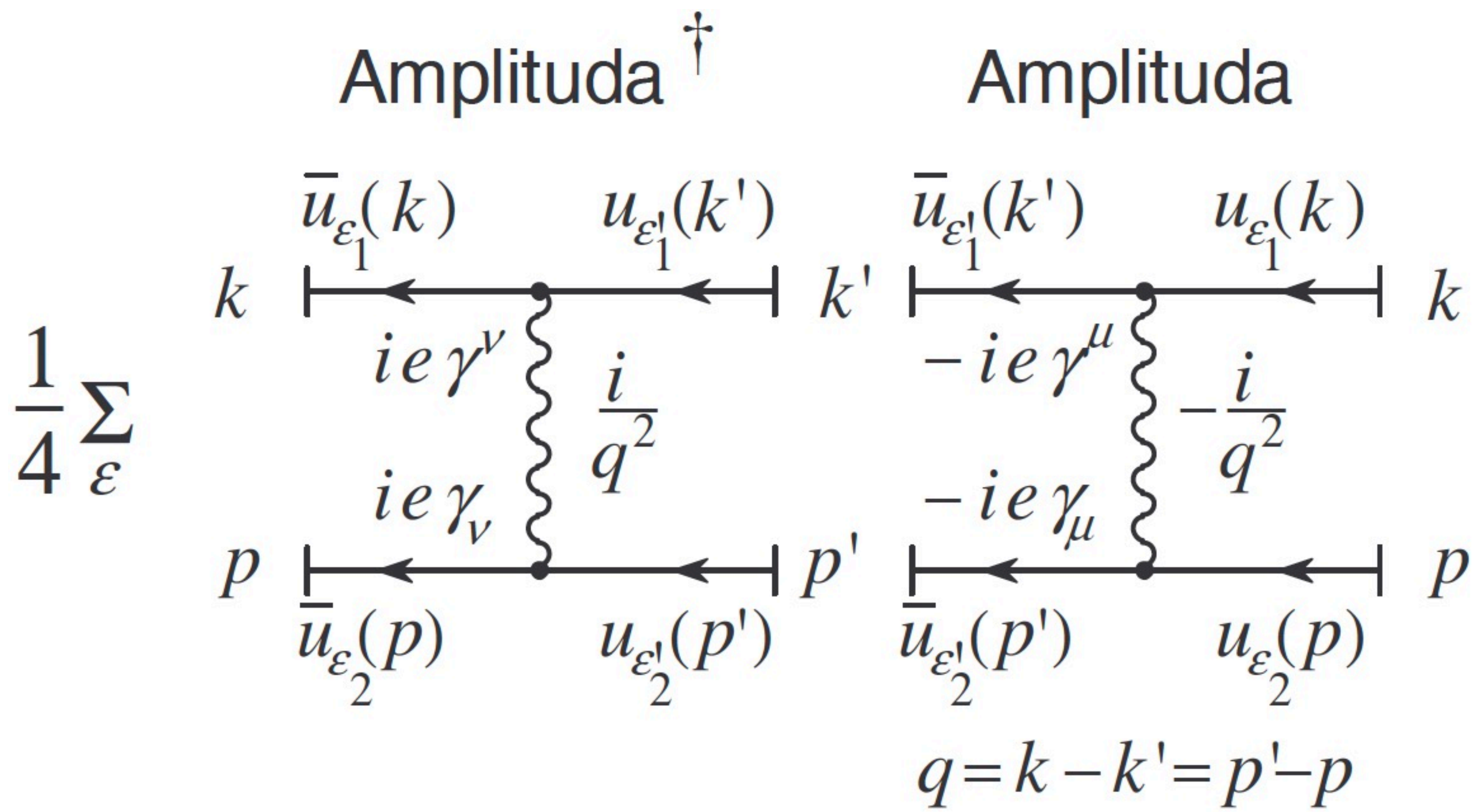
Last time
Elastic cross-section

$$\delta((p+q)^2 - M^2) = \delta(2M\nu - Q^2) = \frac{1}{2M}\delta\left(\nu - \frac{Q^2}{2M}\right)$$

$$\frac{d\sigma}{dQ^2} = \frac{1}{16M^2\omega^2}\frac{1}{4\pi}\int d\nu\,\frac{1}{4}\sum_{\text{pol}}|\mathcal{M}_{fi}|^2\,\delta\left(\nu - \frac{Q^2}{2M}\right)$$

Next: calculate amplitude squared averaged over initial polarizations and summed over final polarizations

Amplitude squared

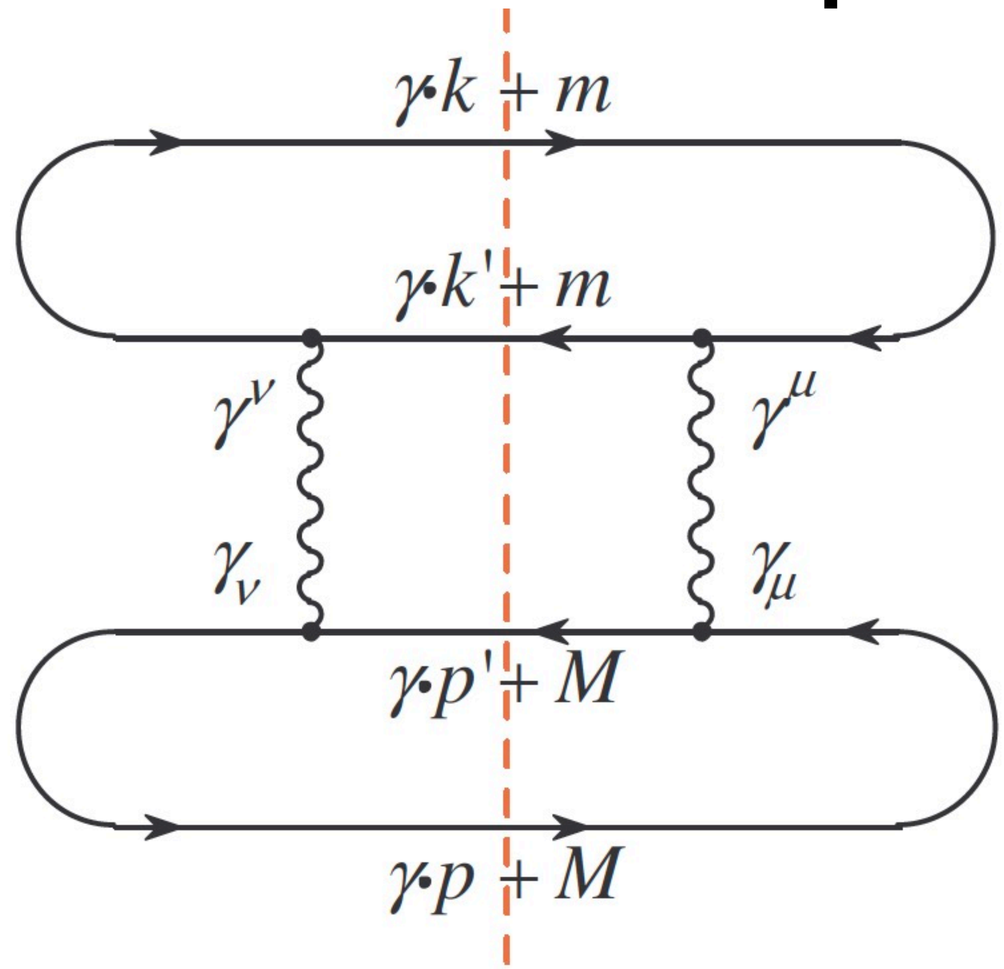


using the following identity identity

$$\sum_{\varepsilon} [u_\varepsilon(p)]_\alpha \, [\overline{u}_\varepsilon(p)]_\beta = (\gamma \cdot p + m)_{\alpha\beta}$$

we get:

Amplitude squared



$$\frac{1}{4}\sum_{\varepsilon} A^{\dagger}A = \frac{e^4 e_M^2}{4q^4} \underbrace{\mathrm{Tr}\left[\gamma^{\mu}(\gamma\cdot k+m)\gamma^{\nu}(\gamma\cdot k'+m)\right]}_{2L^{\mu\nu}(k,k')}$$

$$\underbrace{\mathrm{Tr}\left[\gamma_{\mu}(\gamma\cdot p+M)\gamma_{\nu}(\gamma\cdot p'+M)\right]}_{2L_{\mu\nu}(p,p')} \quad = \frac{e^4 e_M^2}{q^4} L^{\mu\nu}(k,k')L_{\mu\nu}(p,p')$$

Calculating traces

$$L_{\mu\nu}(p,p') = \frac{1}{2}\mathrm{Tr}\left[\gamma_\mu(\gamma\cdot p + M)\gamma_\nu(\gamma\cdot p' + M)\right]$$
$$= 2\left[p_\mu p'_\nu + p'_\mu p_\nu + \frac{q^2}{2}g_{\mu\nu}\right]$$

Gauge invariance

$$q^\mu L_{\mu\nu}(p,p') = q^\nu L_{\mu\nu}(p,p') = 0$$

Check:

$$p\cdot p' = M^2 - q^2/2$$

$$q^\mu L_{\mu\nu} = 2\left[(p'-p)\cdot p\, p'_\nu + (p'-p)\cdot p'\, p_\nu + \frac{q^2}{2}(p'-p)_\nu\right]$$
$$= 2\left[\left(M^2 - \frac{q^2}{2} - M^2\right)p'_\nu + \left(M^2 + \frac{q^2}{2} - M^2\right)p_\nu + \frac{q^2}{2}(p'-p)_\nu\right] = 0$$

Invariant form

$$L_{\mu\nu}(p,q) = 4\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right) - q^2\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)$$

Two structures are separately gauge invariant.

Treat two coefficients as free paramteters

$$L^{\mu\nu}(p,q) = \mathcal{A}\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right) - \mathcal{B}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)$$

and skip q_μ terms because of gauge invariance

$$\rightarrow \mathcal{A}p^\mu p^\nu + \mathcal{B}g^{\mu\nu}$$

where for elastic scattering on an elementary fermion

$$\boxed{\mathcal{A} = 4,\ \mathcal{B} = q^2 = -Q^2}$$

Squared amplitude

$$\frac{1}{4}\sum_{\varepsilon} A^{\dagger}A = \frac{e^4 e_M^2}{q^4}\{\mathcal{A}\, p^{\mu}p^{\nu}L_{\mu\nu}(k,k') + \mathcal{B}\, g^{\mu\nu}L_{\mu\nu}(k,k')\}$$

Compute in our kinematics:

$$p^{\mu}p^{\nu}L_{\mu\nu}(k,k') = 2\left[2\,(p\cdot k)\,(p\cdot k') - \frac{Q^2}{2}M^2\right] = 4M^2\omega\omega'\left(1-\sin^2\frac{\theta}{2}\right) = 4M^2\omega\omega'\cos^2\frac{\theta}{2}$$

$$g^{\mu\nu}L_{\mu\nu}(k,k') = 2\left[2\,(k\cdot k') - 2Q^2\right] = -2Q^2 = -8\omega\omega'\sin^2\frac{\theta}{2}$$

which finally gives

$$\frac{1}{4}\sum_{\varepsilon} A^{\dagger}A = \frac{M^2}{\omega\omega'}\frac{e^4 e_M^2}{\sin^4\frac{\theta}{2}}\left\{\frac{\mathcal{A}}{4}\cos^2\frac{\theta}{2} - \frac{\mathcal{B}}{4M^2}\,2\sin^2\frac{\theta}{2}\right\}$$

Elastic cross-section:

$$\frac{d\sigma}{dQ^2} = \frac{\pi\alpha^2}{4\omega^2 \sin^4\frac{\theta}{2}} \int \frac{e_p^2}{\omega\omega'} \left\{ \frac{\mathcal{A}}{4}\cos^2\frac{\theta}{2} - \frac{\mathcal{B}}{2M^2}\sin^2\frac{\theta}{2} \right\} d\nu\, \delta\left(\nu - \frac{Q^2}{2M}\right)$$

$$= \frac{\pi\alpha^2}{4\omega^2 \sin^4\frac{\theta}{2}} \frac{e_p^2}{\omega\omega'} \left\{ \cos^2\frac{\theta}{2} + \frac{Q^2}{2M^2}\sin^2\frac{\theta}{2} \right\}.$$

Recall:

$$\frac{1}{4}\sum_{\text{pol}} |\mathcal{M}_{fi}|^2 = \frac{e_1^2 e_2^2}{(q^2)^2} L^{\nu\mu}(k,k')\, L_{\nu\mu}(p,p')$$

$$L_{\nu\mu}(p,q) = 4\overbrace{\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)}^{A} + \overbrace{q^2\left(g_{\nu\mu} - \frac{q_\nu q_\mu}{q^2}\right)}^{B}$$

Elastic cross-section:

$$\frac{d\sigma}{dQ^2} = \frac{\pi\alpha^2}{4\omega^2\sin^4\frac{\theta}{2}}\int\frac{e_p^2}{\omega\omega'}\left\{\frac{\mathcal{A}}{4}\cos^2\frac{\theta}{2} - \frac{\mathcal{B}}{2M^2}\sin^2\frac{\theta}{2}\right\}d\nu\,\delta\left(\nu - \frac{Q^2}{2M}\right)$$
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Recall:

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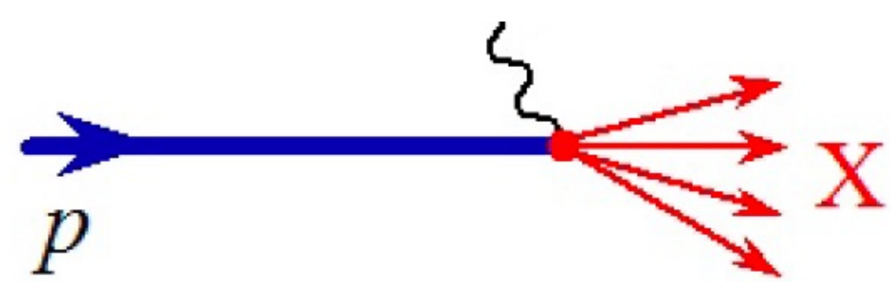
$$L_{\nu\mu}(p,q) = 4\underbrace{\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right)\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)}_{\mathcal{A}} + q^2\underbrace{\left(g_{\nu\mu} - \frac{q_\nu q_\mu}{q^2}\right)}_{\mathcal{B}}$$

Inelastic case:
1) ν not fixed (X not mesured)
2) proton is not elementary



$$W_{\mu\nu}(p,q) = \underbrace{4W_2}_{\mathcal{A}}\left(p_\mu - \frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2}q_\nu\right) + \underbrace{4M^2W_1}_{-\mathcal{B}}\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)$$

Inelastic cross-section:

$$\frac{d\sigma}{dQ^2 d\nu} = \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{ \frac{\mathcal{A}}{4}\cos^2\frac{\theta}{2} - \frac{\mathcal{B}}{4M^2} 2\sin^2\frac{\theta}{2} \right\}$$

$$= \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{ W_2(Q^2,\nu)\cos^2\frac{\theta}{2} + 2W_1(Q^2,\nu)\sin^2\frac{\theta}{2} \right\}$$

Two unknown functions describing the proton structure: W_1 and W_2 depending on two independent variables: Q^2 and ν

Inelastic case:

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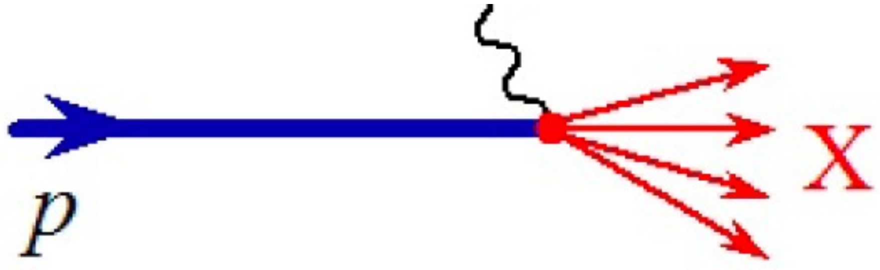


$$W_{\mu\nu}(p,q) = \underbrace{4W_2}_{\mathcal{A}} \left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right) + \underbrace{4M^2 W_1}_{-\mathcal{B}} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)$$

Bjorken Scaling

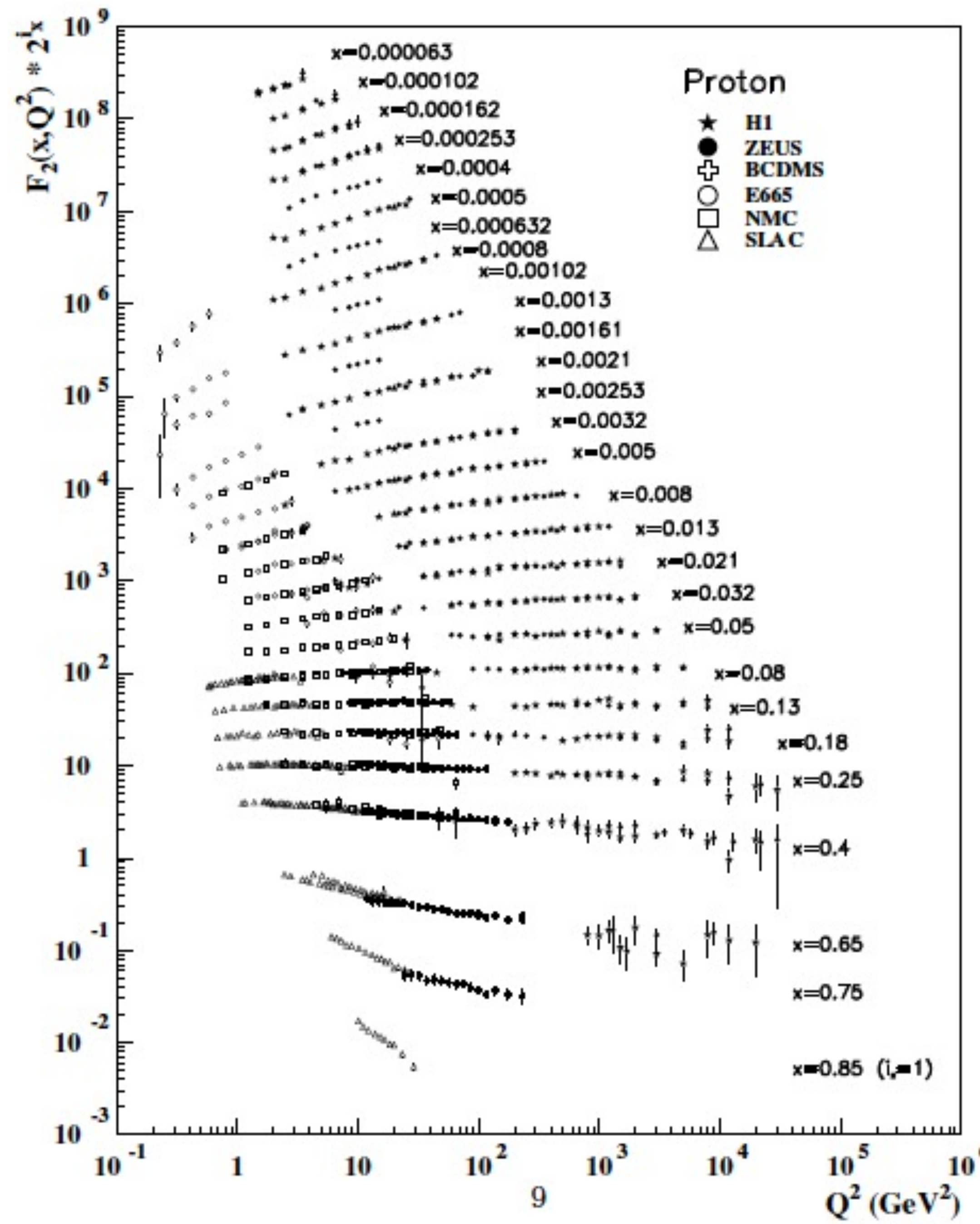
Bjorken limit:

$$Q^2, \nu \to \infty \qquad Q^2/\nu$$

$$MW_1(Q^2, \nu) = F_1(x)$$
$$\nu W_2(Q^2, \nu) = F_2(x)$$

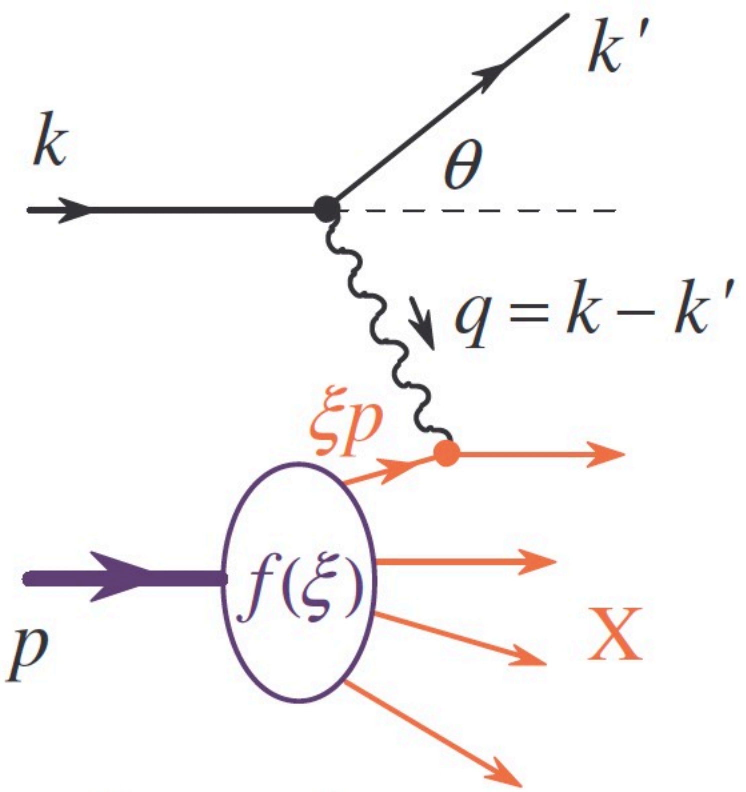
where:

$$x = \frac{Q^2}{2M\nu}$$



Feynman Parton Model

Inelastic scattering on proton
is a sum of elastic scattrings on partons
that are parallel to p
and carry momentum fraction ξ



In the proton rest frame we have to assume that parton mass is

$$m_\xi = \xi M$$

then the on-shell condition for the struck parton reads

$$(\xi p + q)^2 = m_\xi^2$$

$$\xi^2 M^2 + 2\xi M\nu - Q^2 = \xi^2 M^2 \;\rightarrow\; \xi = \frac{Q^2}{2M\nu} = x$$

ξ is the same as Bjorken x !

parton elastic cross-section with proton mas M replaced by $\xi_i M$
and proton charge replaced by parton charge e_i

$$\left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}} = \frac{\pi\alpha^2 e_i^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{\cos^2\frac{\theta}{2} + \frac{Q^2}{4\xi_i^2 M^2} 2\sin^2\frac{\theta}{2}\right\} \delta\left(\nu - \frac{1}{\xi_i}\frac{Q^2}{2M}\right)$$

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multiply by probabilty of finding parton i in the proton,
sum over all partons and integrate over $d\xi_i$ and you get the inelastic cross-section on the proton

$$\frac{d\sigma}{dQ^2 d\nu} = \sum_i \int d\xi_i\, f_i(\xi_i) \left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}}$$

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sum over all partons and integrate over $d\xi_i$ and you get the inelastic cross-section on the proton
expresed in terms of the Bjorken functions $W_{1,2}$

$$\frac{d\sigma}{dQ^2 d\nu} = \sum_i \int d\xi_i\, f_i(\xi_i) \left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}} = \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{W_2 \cos^2\frac{\theta}{2} + 2W_1 \sin^2\frac{\theta}{2}\right\}$$

parton elastic cross-section with proton mas M replaced by $\xi_i M$
and proton charge replaced by parton charge e_i

$$\left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}} = \frac{\pi\alpha^2 e_i^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}}\left\{\cos^2\frac{\theta}{2} + \frac{Q^2}{4\xi_i^2 M^2} 2\sin^2\frac{\theta}{2}\right\} \delta\left(\nu - \frac{1}{\xi_i}\frac{Q^2}{2M}\right)$$

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expresed in terms of the Bjorken functions $W_{1,2}$

$$\frac{d\sigma}{dQ^2 d\nu} = \sum_i \int d\xi_i\, f_i(\xi_i) \left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}} = \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}}\left\{W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2}\right\}$$

we can now immediately calculate $W_{1,2}$ in terms of $f(\xi)$

$$W_2 = \sum_i e_i^2 \int d\xi\, f_i(\xi)\delta\left(\nu - \nu\frac{x}{\xi}\right) = \sum_i e_i^2 \int d\xi\, f_i(\xi)\frac{\xi^2}{\nu x}\delta\left(\xi - x\right) = \frac{1}{\nu}\sum_i e_i^2 x\, f_i(x)$$

$$W_1 = \sum_i e_i^2 \int d\xi\, f_i(\xi)\frac{Q^2}{4\xi^2 M^2}\frac{\xi^2}{\nu x}\delta\left(\xi - x\right) = \frac{1}{2M}\sum_i e_i^2\, f_i(x). \qquad x = \frac{Q^2}{2M\nu}$$

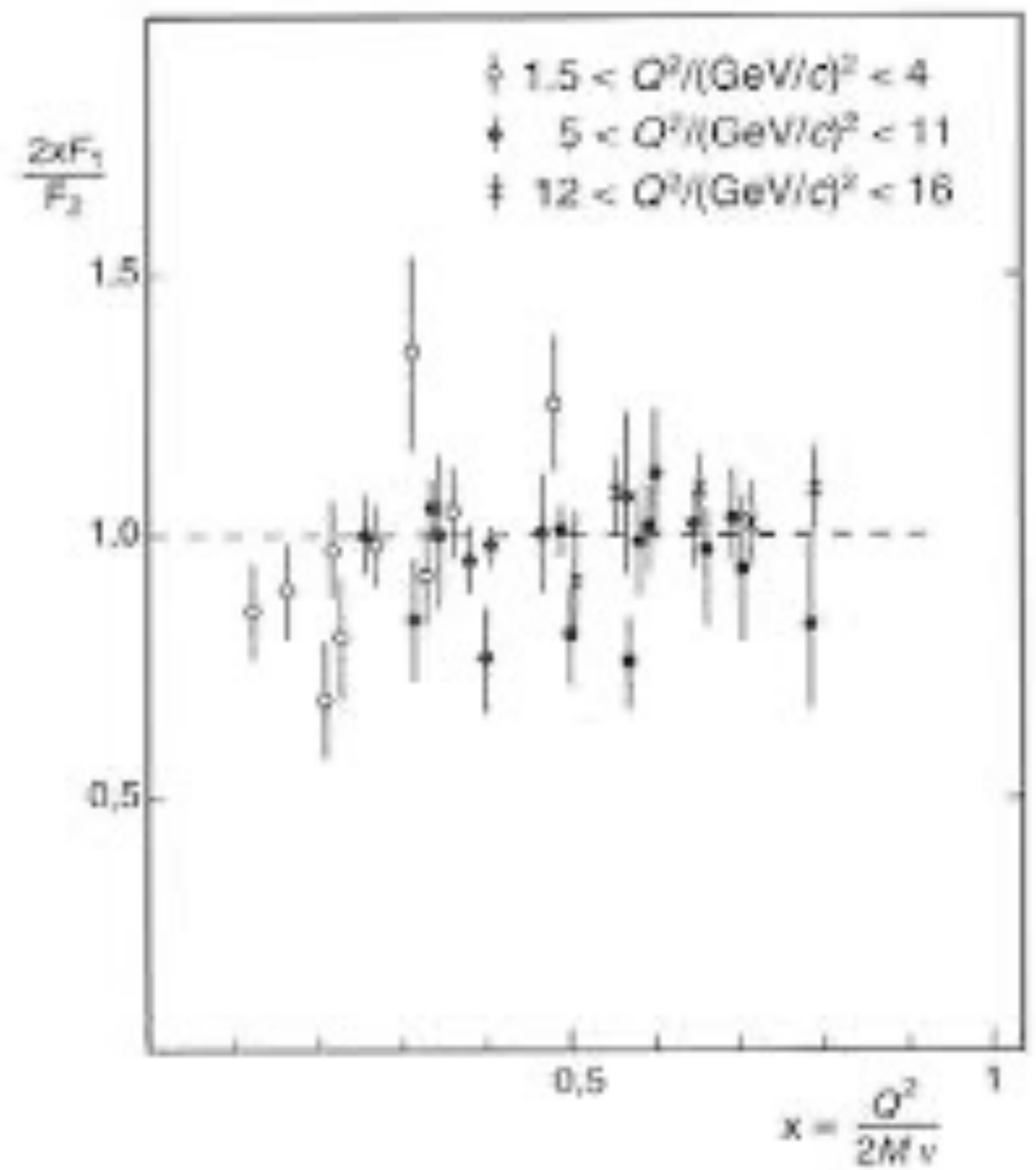
Bjorken Scaling vs. Parton Model

$$F_2(x) = \nu W_2 = x \sum_i e_i^2 f_i(x)$$

$$F_1(x) = M W_1 = \frac{1}{2} \sum_i e_i^2 \, f_i(x)$$

$$F_2(x) = 2x F_1(x)$$

in parton model structure functions are related: Callan-Gross relation



Quarks as Partons

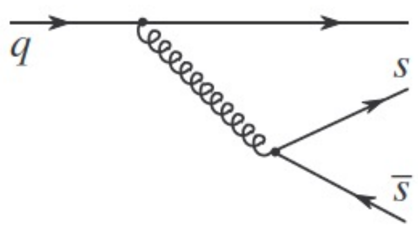
$$F_2^{\mathrm{p}}(x) = \frac{4}{9}x\left[u_{\mathrm{p}}(x) + \overline{u}_{\mathrm{p}}(x)\right] + \frac{1}{9}x\left[d_{\mathrm{p}}(x) + \overline{d}_{\mathrm{p}}(x) + s_{\mathrm{p}}(x) + \overline{s}_{\mathrm{p}}(x)\right]$$

$$F_2^{\mathrm{n}}(x) = \frac{4}{9}x\left[u_{\mathrm{n}}(x) + \overline{u}_{\mathrm{n}}(x)\right] + \frac{1}{9}x\left[d_{\mathrm{n}}(x) + \overline{d}_{\mathrm{n}}(x) + s_{\mathrm{n}}(x) + \overline{s}_{\mathrm{n}}(x)\right]$$

assuming isospin symmetry:

$$u_{\mathrm{p}} = d_{\mathrm{n}} = u, \quad d_{\mathrm{p}} = u_{\mathrm{n}} = d, \quad s_{\mathrm{p}} = s_{\mathrm{n}} = s$$

no strangeness in the nucleon: $\int dx(s(x) - \overline{s}(x)) = 0$



Quarks as Partons

proton and neutron charges

$$q_{\mathrm{p}} = \int dx \left[\frac{2}{3}(u(x) - \overline{u}(x)) - \frac{1}{3}(d(x) - \overline{d}(x)) - \frac{1}{3}(s(x) - \overline{s}(x))\right] = 1$$

$$\updownarrow = 0$$

$$q_{\mathrm{n}} = \int dx \left[\frac{2}{3}(d(x) - \overline{d}(x)) - \frac{1}{3}(u(x) - \overline{u}(x)) - \frac{1}{3}(s(x) - \overline{s}(x))\right] = 0$$

imply constraints on the parton distributions (PDF's):

$$\int dx(u(x) - \overline{u}(x)) = 2, \quad \int dx(d(x) - \overline{d}(x)) = 1, \quad \int dx(s(x) - \overline{s}(x)) = 0$$

valence and sea quarks: $u = u_v + q_s, \quad d = d_v + q_s, \quad \overline{u} = \overline{d} = \overline{s} = s = q_s$

total momenum – for typical parametrizations

$$\int dx\, x(u(x) + \overline{u}(x) + d(x) + \overline{d}(x) + s(x) + \overline{s}(x)) = 1 - \varepsilon$$

$\varepsilon \sim 45\%$

there must be other partons that do not inteact electromagnetically: gluons