QCD lecture 15c

January 25

Nonlinear realization of SU(*N*) x SU(*N*)

We can parametrize SU(*N*) matrix as $U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)$

where for SU(2)

$$\phi(x) = \sum_{i=1}^{3} \tau_i \phi_i(x) = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} \equiv \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$$

or for SU(3)

$$\phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) = \begin{pmatrix} \phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & -\phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix}$$

$$\equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix},$$

[there exist different conventions for signs of particle fields]

Nonlinear realization of SU(*N*) x SU(*N*)

Define $M_3 \equiv \left\{ U : M^4 \to \mathrm{SU}(N) | U(x) = \exp\left(i\frac{\phi(x)}{F_0} \right) \right\}$

The homomorphism

$$\varphi : G \times M_3 \to M_3 \quad \text{with} \quad \varphi[(L,R),U](x) \equiv RU(x)L^\dagger$$

defines an operation of G on M_3

1. $RUL^\dagger \in M_3$, since $U \in M_3$ and $R, L^\dagger \in \mathrm{SU}(N)$.
2. $\varphi[(1_{N\times N}, 1_{N\times N}), U](x) = 1_{N\times N} U(x) 1_{N\times N} = U(x)$.
3. Let $g_i = (L_i, R_i) \in G$ and thus $g_1 g_2 = (L_1 L_2, R_1 R_2) \in G$.

$$\begin{aligned} \varphi[g_1, \varphi[g_2, U]](x) &= \varphi[g_1, (R_2 U L_2^\dagger)](x) = R_1 R_2 U(x) L_2^\dagger L_1^\dagger, \\ \varphi[g_1 g_2, U](x) &= R_1 R_2 U(x)(L_1 L_2)^\dagger = R_1 R_2 U(x) L_2^\dagger L_1^\dagger. \end{aligned}$$

all group requirements are fulfilled. This mapping is called nonlinear because M_3 is not a vector space (sum of two *U* matrices is not a unitary matrix).

Nonlinear realization of SU(*N*) x SU(*N*)

The origin (vacuum) corresponds to $\phi(x) = 0$, i.e. $U_0 = 1$

Indeed
$$\begin{aligned} \varphi[g = (V, V), 1] &= VV^\dagger = 1 \\ \varphi[g = (A, A^\dagger), 1] &= A^\dagger A^\dagger \neq 1 \end{aligned}$$

Axial symmetry is broken, left and right fermions must be transformed the same way.

Transformation of fieds $\phi(x)$
$$U = 1 + i\frac{\phi}{F_0} - \frac{\phi^2}{2F_0^2} + \cdots$$

and transformation matrix $V = \exp\left(-i\Theta_a^V \frac{\lambda_a}{2}\right)$ give

$$\phi = \lambda_b \phi_b \overset{h \in \mathrm{SU}(3)_V}{\mapsto} V\phi V^\dagger = \phi - i\Theta_a^V \underbrace{[\frac{\lambda_a}{2}, \phi_b \lambda_b]}_{\phi_b i f_{abc} \lambda_c} + \cdots = \phi + f_{abc}\Theta_a^V \phi_b \lambda_c + \cdots$$

Fields $\phi(x)$ transform according to the adjoint rep. of SU(3) (like gaue fields...)

Effective lagrangian

Matrix U is our "building block". Langrangian must be symmetric under global

$$\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V \qquad U(x) \mapsto RU(x)L^\dagger \quad U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)$$

The most general lagrangian with two derivatives (Weinberg lagrangian)

$$\mathcal{L}_{\mathrm{eff}} = \frac{F_0^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)$$

where (experimentally) $F_0 \approx 93\ \mathrm{MeV}$ can be deduced from $\pi^+ \to \mu^+ \nu_\mu$

Invariance:

$$U \mapsto RUL^\dagger \quad \partial_\mu U \mapsto R\partial_\mu U L^\dagger \quad U^\dagger \mapsto LU^\dagger R^\dagger \quad \partial_\mu U^\dagger \mapsto L\partial_\mu U^\dagger R^\dagger$$

$$\mathcal{L}_{\mathrm{eff}} \mapsto \frac{F_0^2}{4}\mathrm{Tr}\Big(R\partial_\mu U \underbrace{L^\dagger L}_{1} \partial^\mu U^\dagger R^\dagger\Big) = \frac{F_0^2}{4}\mathrm{Tr}\Big(\underbrace{R^\dagger R}_{1} \partial_\mu U \partial^\mu U^\dagger\Big) = \mathcal{L}_{\mathrm{eff}}$$

Effective lagrangian

Expanding $U = 1 + i\phi/F_0 + \cdots \qquad \partial_\mu U = i\partial_\mu\phi/F_0 + \cdots$

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4}\text{Tr}\left[\frac{i\partial_\mu\phi}{F_0}\left(-\frac{i\partial^\mu\phi}{F_0}\right)\right] + \cdots = \frac{1}{4}\text{Tr}(\lambda_a\partial_\mu\phi_a\lambda_b\partial^\mu\phi_b) + \cdots$$

$$= \frac{1}{4}\partial_\mu\phi_a\partial^\mu\phi_b\text{Tr}(\lambda_a\lambda_b) + \cdots = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + \mathcal{L}_{\text{int}}$$

we get usual lagrangian plus interactions that proceed only through derivatives (momenta). For small momenta higher derivative terms are small. Interactions are even in ϕ_a Parity

$\phi_a(\vec{x},t) \mapsto -\phi_a(-\vec{x},t) \qquad U(\vec{x},t) \mapsto U^\dagger(-\vec{x},t)$

This lagrangian is unique up to total derivatives. E.g.:

$$\text{Tr}[(\partial_\mu\partial^\mu U)U^\dagger] = \partial_\mu[\text{Tr}(\partial^\mu U U^\dagger)] - \text{Tr}(\partial^\mu U\partial_\mu U^\dagger)$$

Single derivatives vanish under trace $\text{Tr}(\partial_\mu U U^\dagger) = 0$

Currents

Left currents. Set $\Theta_a^R = 0$ and make left transformation space-time dependent:

$\Theta_a^L = \Theta_a^L(x)$

Then $$U \mapsto U' = RUL^\dagger = U\left(1 + i\Theta_a^L \frac{\lambda_a}{2}\right)$$

$$\partial_\mu U \mapsto \partial_\mu U' = \partial_\mu U \left(1 + i\Theta_a^L \frac{\lambda_a}{2}\right) + U i \partial_\mu \Theta_a^L \frac{\lambda_a}{2}$$

$$\partial_\mu U^\dagger \mapsto \partial_\mu U'^\dagger = \left(1 - i\Theta_a^L \frac{\lambda_a}{2}\right) \partial_\mu U^\dagger - i\partial_\mu \Theta_a^L \frac{\lambda_a}{2} U^\dagger$$

and:

$$\begin{aligned} \delta\mathcal{L}_{\text{eff}} &= \frac{F_0^2}{4} \text{Tr}\left[U i\partial_\mu \Theta_a^L \frac{\lambda_a}{2} \partial^\mu U^\dagger + \partial_\mu U \left(-i\partial^\mu \Theta_a^L \frac{\lambda_a}{2} U^\dagger\right)\right] \\ &= \frac{F_0^2}{4} i\partial_\mu \Theta_a^L \text{Tr}\left[\frac{\lambda_a}{2}(\partial^\mu U^\dagger U - U^\dagger \partial^\mu U)\right] \\ &= \frac{F_0^2}{4} i\partial_\mu \Theta_a^L \text{Tr}\left(\lambda_a \partial^\mu U^\dagger U\right). \end{aligned}$$

⟵ $\partial^\mu U^\dagger U = -U^\dagger \partial^\mu U$

Left current:

$$J_L^{\mu,a} = \frac{\partial \delta\mathcal{L}_{\text{eff}}}{\partial \partial_\mu \Theta_a^L} = i\frac{F_0^2}{4} \text{Tr}\left(\lambda_a \partial^\mu U^\dagger U\right)$$

Currents

Left currents. Set $\Theta_a^R = 0$ and make left transformation space-time dependent:

$\Theta_a^L = \Theta_a^L(x)$

Then
$$U \mapsto U' = RUL^\dagger = U\left(1 + i\Theta_a^L \frac{\lambda_a}{2}\right)$$

$$\partial_\mu U \mapsto \partial_\mu U' = \partial_\mu U\left(1 + i\Theta_a^L \frac{\lambda_a}{2}\right) + Ui\partial_\mu \Theta_a^L \frac{\lambda_a}{2}$$

$$\partial_\mu U^\dagger \mapsto \partial_\mu U'^\dagger = \left(1 - i\Theta_a^L \frac{\lambda_a}{2}\right)\partial_\mu U^\dagger - i\partial_\mu \Theta_a^L \frac{\lambda_a}{2} U^\dagger$$

and:
$$\begin{aligned}\delta\mathcal{L}_{\text{eff}} &= \frac{F_0^2}{4}\text{Tr}\left[Ui\partial_\mu\Theta_a^L\frac{\lambda_a}{2}\partial^\mu U^\dagger + \partial_\mu U\left(-i\partial^\mu\Theta_a^L\frac{\lambda_a}{2}U^\dagger\right)\right] \\ &= \frac{F_0^2}{4} i\partial_\mu\Theta_a^L \text{Tr}\left[\frac{\lambda_a}{2}(\partial^\mu U^\dagger U - U^\dagger\partial^\mu U)\right] \\ &= \frac{F_0^2}{4} i\partial_\mu\Theta_a^L \text{Tr}\left(\lambda_a \partial^\mu U^\dagger U\right).\end{aligned}$$

⬅ $\partial^\mu U^\dagger U = -U^\dagger \partial^\mu U$

Left current:
$$J_L^{\mu,a} = \frac{\partial\delta\mathcal{L}_{\text{eff}}}{\partial\partial_\mu\Theta_a^L} = i\frac{F_0^2}{4}\text{Tr}\left(\lambda_a\partial^\mu U^\dagger U\right)$$

Right current:
$$J_R^{\mu,a} = \frac{\partial\delta\mathcal{L}_{\text{eff}}}{\partial\partial_\mu\Theta_a^R} = -i\frac{F_0^2}{4}\text{Tr}\left(\lambda_a U\partial^\mu U^\dagger\right)$$

Currents

We can now calculate vector and axial currents:

$$J_V^{\mu,a} = J_R^{\mu,a} + J_L^{\mu,a} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a[U,\partial^\mu U^\dagger]\right),$$

$$J_A^{\mu,a} = J_R^{\mu,a} - J_L^{\mu,a} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a\{U,\partial^\mu U^\dagger\}\right)$$

Internal parity:

$$J_V^{\mu,a} \overset{\phi \mapsto -\phi}{\mapsto} -i\frac{F_0^2}{4}\mathrm{Tr}[\lambda_a(U^\dagger\partial^\mu U - \partial^\mu U U^\dagger)]$$

$\partial^\mu U^\dagger U = -U^\dagger \partial^\mu U$ ⇒

$$= -i\frac{F_0^2}{4}\mathrm{Tr}[\lambda_a(-\partial^\mu U^\dagger U + U\partial^\mu U^\dagger)] = J_V^{\mu,a}$$

$$J_A^{\mu,a} \overset{\phi \mapsto -\phi}{\mapsto} -i\frac{F_0^2}{4}\mathrm{Tr}[\lambda_a(U^\dagger\partial^\mu U + \partial^\mu U U^\dagger)]$$

$$= i\frac{F_0^2}{4}\mathrm{Tr}[\lambda_a(\partial^\mu U^\dagger U + U\partial^\mu U^\dagger)] = -J_A^{\mu,a}$$

Matrix elemen of axial current

Axial current $J_A^{\mu,a} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a\{U, \partial^\mu U^\dagger\}\right)$

expanding: $J_A^{\mu,a} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a\left\{1+\cdots, -i\frac{\lambda_b\partial^\mu\phi_b}{F_0}+\cdots\right\}\right) = -F_0\partial^\mu\phi_a + \cdots$

Matrix element of axial current between GB and vacuum:

$$\begin{aligned}
\langle 0|\, J_A^{\mu,a}(x)\,\left|\phi^b(p)\right\rangle &= -F_0\,\langle 0|\,\partial^\mu\phi^a(x)\,\left|\phi^b(p)\right\rangle \\
&= -F_0\int\frac{d^4p'}{(2\pi)^4}\partial^\mu e^{-ip\cdot x}\underbrace{\langle 0|\,\phi^a(p')\,\left|\phi^b(p)\right\rangle}_{=(2\pi)^4\delta^{(4)}(p'-p)\,\delta^{ab}} \\
&= ip^\mu e^{-ip\cdot x}F_0\delta^{ab}.
\end{aligned}$$

This agrees with previous result from QCD

$$\langle 0|A_\mu^a(0)|\phi^b(p)\rangle = ip_\mu F_0\delta^{ab}$$

Mass term

In QCD

$$\mathcal{L}_M = -\bar{q}_R M q_L - \bar{q}_L M^\dagger q_R, \quad M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

This would be invariant if $M \mapsto RML^\dagger$

What is the effective lagrangian that respects this would be symmetry? To the lowest order in M

$$\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2} \mathrm{Tr}(MU^\dagger + UM^\dagger)$$

where B_0 is a new parameter. This means that the ground state $(U = 1)$ energy density is

$$\langle \mathcal{H}_{\text{eff}} \rangle = -F_0^2 B_0 (m_u + m_d + m_s)$$

In QCD

$$\left. \frac{\partial \langle 0 | \mathcal{H}_{\text{QCD}} | 0 \rangle}{\partial m_q} \right|_{m_u = m_d = m_s = 0} = \frac{1}{3} \langle 0 | \bar{q} q | 0 \rangle_0 = \frac{1}{3} \langle \bar{q} q \rangle$$

and we have

$$3 F_0^2 B_0 = -\langle \bar{q} q \rangle$$

Mass term

$$\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2} \mathrm{Tr}(MU^\dagger + UM^\dagger) \qquad 3F_0^2 B_0 = -\langle \bar{q} q \rangle$$

Constant B_0 has dimension 1 (energy).

Expanding $U(x) \;=\; \exp\left(i\frac{\phi(x)}{F_0}\right)$ gives $\mathcal{L}_{\text{s.b}} = -\frac{B_0}{2}\mathrm{Tr}(\phi^2 M) + \cdots$

Using

$$\phi(x) \;=\; \sum_{a=1}^{8} \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

one gets

$$\begin{aligned} \mathrm{Tr}(\phi^2 M) \;=\; & 2(m_u + m_d)\pi^+\pi^- + 2(m_u + m_s)K^+K^- + 2(m_d + m_s)K^0\bar{K}^0 \\ & +(m_u + m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u - m_d)\pi^0\eta + \frac{m_u + m_d + 4m_s}{3}\eta^2. \end{aligned}$$

mixing

Mass term

$$\mathrm{Tr}(\phi^2 M) = 2(m_u+m_d)\pi^+\pi^- + 2(m_u+m_s)K^+K^- + 2(m_d+m_s)K^0\bar{K}^0 + (m_u+m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u-m_d)\pi^0\eta + \frac{m_u+m_d+4m_s}{3}\eta^2.$$

Isospin symmetric limit $m_u = m_d = m$

$\mathcal{L}_{\mathrm{s.b}} = -\frac{B_0}{2}\mathrm{Tr}(\phi^2 M)$ implies the following meson masses

$$M_\pi^2 = 2B_0 m,$$
$$M_K^2 = B_0(m+m_s),$$
$$M_\eta^2 = \frac{2}{3}B_0\,(m+2m_s)$$

where $B_0 = -\langle\bar{q}q\rangle/(3F_0^2)$

Gell-Mann – Okubo mass relation (does not depend on B_0)

$$4M_K^2 = 4B_0(m+m_s) = 2B_0(m+2m_s) + 2B_0 m = 3M_\eta^2 + M_\pi^2$$

$$L = 4\times 494^2 = 976\,144\,\mathrm{MeV}^2 \quad R = 3\times 548^2 + 138^2 = 919\,956\,\mathrm{MeV}^2$$

Mass term

$$\mathrm{Tr}(\phi^2 M) = 2(m_u+m_d)\pi^+\pi^- + 2(m_u+m_s)K^+K^- + 2(m_d+m_s)K^0\bar{K}^0 + (m_u+m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u-m_d)\pi^0\eta + \frac{m_u+m_d+4m_s}{3}\eta^2.$$

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$\mathcal{L}_{\mathrm{s.b}} = -\frac{B_0}{2}\mathrm{Tr}(\phi^2 M)$ implies the following meson masses

$$\begin{aligned} M_\pi^2 &= 2B_0 m, \\ M_K^2 &= B_0(m+m_s), \\ M_\eta^2 &= \frac{2}{3}B_0\,(m+2m_s) \end{aligned}$$

where $B_0 = -\langle \bar{q}q\rangle/(3F_0^2)$

Gell-Mann – Okubo mass relation (does not depend on B_0)

$$\frac{L-R}{L+R} = 3\%$$

$$4M_K^2 = 4B_0(m+m_s) = 2B_0(m+2m_s) + 2B_0 m = 3M_\eta^2 + M_\pi^2$$

$$L = 4\times 494^2 = 976\,144\,\mathrm{MeV}^2 \quad R = 3\times 548^2 + 138^2 = 919\,956\,\mathrm{MeV}^2$$

PCAC
partially conserved axial current

Let's calculate

$$\begin{aligned}
\langle 0|\,\phi^a(x)\,|\phi^b(p)\rangle &= \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_k}}e^{-ik\cdot x}\,\langle 0|\,a_a(\boldsymbol{k})\sqrt{(2\pi)^3 2E_p}a_b^\dagger(\boldsymbol{p})\,|0\rangle \\
&= \int d^3k\sqrt{\frac{E_p}{E_k}}e^{-ik\cdot x}\,\langle 0|\,a_a(\boldsymbol{k})a_b^\dagger(\boldsymbol{p})\,|0\rangle \\
&= \delta^{ab}e^{-ip\cdot x}
\end{aligned}$$

Every field that has this property is called *interpolating field*.
Let's consider isospin subgroup of SU(3). Axial current matrix element

$$\langle 0|A_i^\mu(x)|\pi_j(q)\rangle = iq^\mu F_0 e^{-iq\cdot x}\delta_{ij}$$

Let's take its divergence

$$\langle 0|\partial_\mu A_i^\mu(x)|\pi_j(q)\rangle = iq^\mu F_0\partial_\mu e^{-iq\cdot x}\delta_{ij} = M_\pi^2 F_0 e^{-iq\cdot x}\delta_{ij} = 2m_q B_0 F_0 e^{-iq\cdot x}\delta_{ij}$$

This means that divergence of the axial current, up to a constant, is itself pion interpolating field. On the other hand $\partial_\mu A_i^\mu = i\,m_q\,(\bar{q}\tau_i\gamma_5 q) = m_q P_i$
so pseudoscalar density is also a pion interpolating field.

PCAC
partially conserved axial current

Let's calculate

$$\begin{aligned} \langle 0|\,\phi^a(x)\,|\phi^b(p)\rangle &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}e^{-ik\cdot x}\,\langle 0|\,a_a(\boldsymbol{k})\sqrt{(2\pi)^3 2E_p}a_b^\dagger(\boldsymbol{p}) \\ &= \int d^3k\sqrt{\frac{E_p}{E_k}}e^{-ik\cdot x}\,\langle 0|\,a_a(\boldsymbol{k})a_b^\dagger(\boldsymbol{p})\,|0\rangle \\ &= \delta^{ab}e^{-ip\cdot x} \end{aligned}$$

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Let's consider isospin subgroup of SU(3). Axial current matrix element

$$\langle 0|A_i^\mu(x)|\pi_j(q)\rangle = iq^\mu F_0 e^{-iq\cdot x}\delta_{ij}$$

$$-\frac{2}{3}\frac{m_q\,\langle\bar{q}q\rangle}{F_0}$$

Let's take its divergence

$$\langle 0|\partial_\mu A_i^\mu(x)|\pi_j(q)\rangle = iq^\mu F_0\partial_\mu e^{-iq\cdot x}\delta_{ij} = M_\pi^2 F_0 e^{-iq\cdot x}\delta_{ij} = 2m_q B_0 F_0 e^{-iq\cdot x}\delta_{ij}$$

This means that divergence of the axial current, up to a constant, is itself pion interpolating field. On the other hand $\partial_\mu A_i^\mu = i\,m_q\,(\,\bar{q}\tau_i\gamma_5 q) = m_q P_i$
so pseudoscalar density is also a pion interpolating field.

Chiral lagrangian

$$\mathcal{L} = \frac{F_0^2}{4}\,\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{F_0^2 B_0}{2}\,\mathrm{Tr}\left(M U^\dagger + U M\right) \qquad U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)$$

Chiral lagrangian is expressed in terms of a U field

$$\phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

- Nonzero quark condensate in chiral limit is a sufficient cond. for a spotaneus χSB
- Quark mass term gives masses to GBs
- Gell-Mann – Okubo mass formula emerges – satisfied experimentally
- Terms with more derivatives and with higher powers of M are possible
- Such theory is not renormalizable, but there is a method to make it predictive: chiral perturbation theory
- Coupling to photons, W and Z by covariant derivatives

Chiral perturbation theory

Effective lagrangian

$$U = \exp\left(i\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\phi}}{F}\right)$$

$$\mathcal{L}_2 = \frac{F^2}{4}\,\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)$$

up to 4 fields in SU(2)

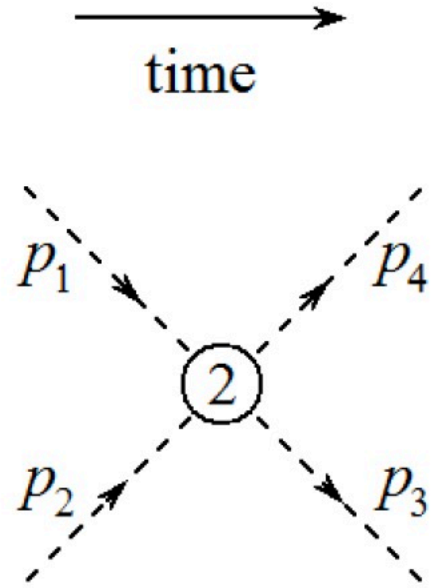
$$\mathcal{L}_2^{(4)} = \frac{1}{6F^2}\left\{(\partial_\mu\boldsymbol{\phi}\cdot\boldsymbol{\phi})\,(\partial^\mu\boldsymbol{\phi}\cdot\boldsymbol{\phi}) - (\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi})\,(\boldsymbol{\phi}\cdot\boldsymbol{\phi})\right\}.$$

Terms with more derivatives (adding weak and elm. interactions via covariant derivative)

$$\begin{aligned}\mathcal{L}_4 \;=\;& L_1\left\{\mathrm{Tr}\left(\partial_\mu U\partial^\mu U^\dagger\right)\right\}^2 + L_2\,\mathrm{Tr}\left(\partial_\mu U\partial_\nu U^\dagger\right)\mathrm{Tr}\left(\partial^\mu U\partial^\nu U^\dagger\right)\\ &+ L_3\,\mathrm{Tr}\left(\partial_\mu U\partial^\mu U^\dagger\partial_\nu U\partial^\nu U^\dagger\right) + \ldots\end{aligned}$$

Coefficients L_i are free, have to be extracted from data.

Weinberg counting – tree level



Scattering from 2-derivative term

$$\mathcal{M}\left(p_1, p_2, p_3, p_4\right) \sim \left(p_1 + p_2\right) \cdot \left(p_3 + p_4\right) - p_1 \cdot p_2 - p_3 \cdot p_4.$$

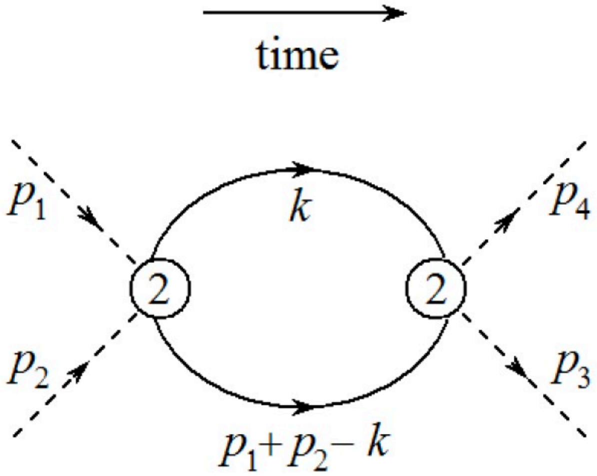
rescale all momenta:

$$p \to tp$$

$$\mathcal{M}\left(p_1, p_2, p_3, p_4\right) \to t^2 \mathcal{M}\left(p_1, p_2, p_3, p_4\right)$$

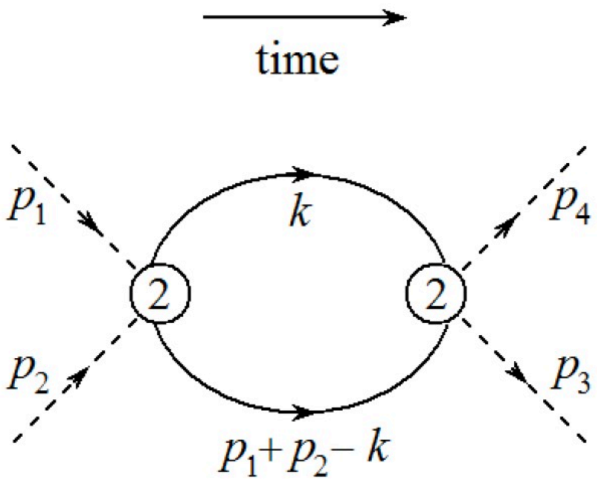
Weinberg counting – one loop

logarithmically divergent



$$\mathcal{M}_{\text{loop}} \sim \int d^4k \, [(p_1+p_2)\cdot(p_1+p_2) - p_1\cdot p_2 - (p_1+p_2-k)\cdot k] \frac{1}{k^2-m^2}\frac{1}{(p_1+p_2-k)^2-m^2} [(p_1+p_2)\cdot(p_3+p_4) - (p_1+p_2-k)\cdot k - p_3\cdot p_4]$$

Weinberg counting – one loop



logarithmically divergent

rescale $\quad p_i \to t p_i,\ m \to t m,\ k \to t q$

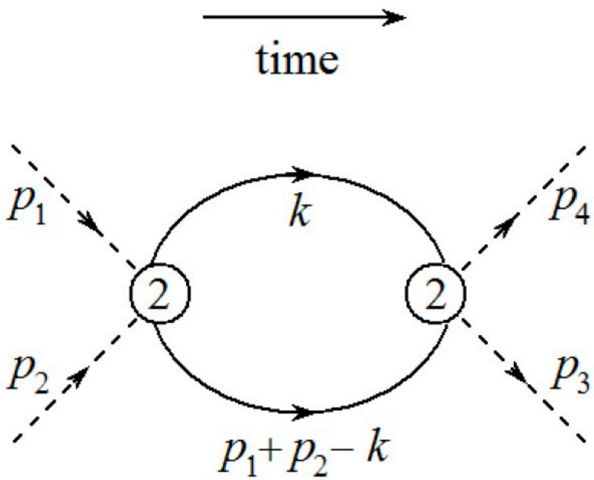
$$\mathcal{M}_{\text{loop}} \sim \overbrace{t^4}^{\text{vertices}}\ \overbrace{t^{-4}}^{\text{props}}\ \overbrace{t^4}^{\text{integration}}$$

$$\int d^4 l\ [(p_1+p_2)\cdot(p_1+p_2) - p_1\cdot p_2 - (p_1+p_2-l)\cdot l]$$

$$\frac{1}{k^2-m^2}\frac{1}{(p_1+p_2-l)^2-m^2}$$

$$[(p_1+p_2)\cdot(p_3+p_4) - (p_1+p_2-k)\cdot l - p_3\cdot p_4]$$

Weinberg counting – one loop

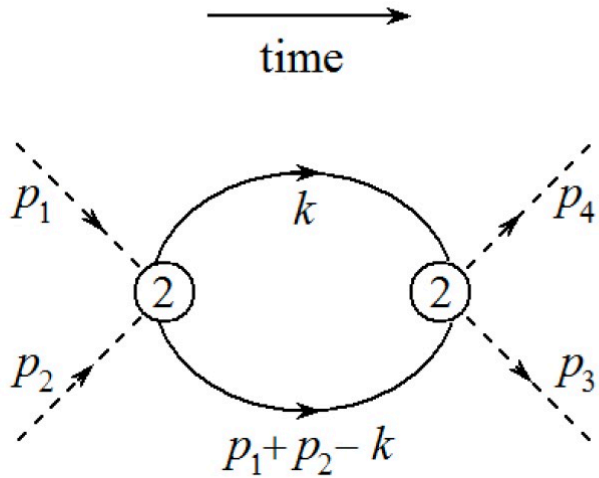


logarithmically divergent

rescale $\quad p_i \to tp_i, \; m \to tm, \; k \to tq$

$$\mathcal{M}_{\text{loop}} \to t^4 \mathcal{M}_{\text{loop}}$$

Weinberg counting – one loop



logarithmically divergent

$\mathcal{M}_{\text{loop}} \to t^4 \mathcal{M}_{\text{loop}}$

This means that the divergence enters at the level of four derivative terms. So it "renormalizes" coefficients L_i rather than the coefficeints following from $\mathcal{L}_2$. Hence we can absorb these divergences to unrenormalized (bare) constants $L_i \to L_i^r$. When we calculate loop corrections involving vertices from $\mathcal{L}_4$ (*i.e.* involving renormalized constants L_i^r) new divernces appear, but they affect some new couplings with higher number of derivatives, but not L_i^r's themselves. This theory is opperative at low energies, so we typcyally stop at the four derivative level (corresponding to p^4). This scheme is known as *chiral perturbation theory* and the resacling procedure introduced above is known as *Weinberg power counting*.