QCD lecture 15b

January 18

Goldstone bosons

$$\langle 0| i[Q_a^A(t), P_a] |0\rangle \;\; = \frac{i}{2} \lim_{p^0 \to 0} \sum_b \left\{ \frac{\langle 0| A_a^0 |\phi^b\rangle}{p^0} \langle \phi^b| P_a |0\rangle - \langle 0| P_a |\phi^b\rangle \frac{\langle \phi^b| A_a^0 |0\rangle}{p^0} \right\}$$

From hermicity and Lorentz invariance $\langle 0| A_a^\mu |\phi^b(p)\rangle = ip^\mu F_\phi \delta^{ab}$

and we get

$$\langle 0| i[Q_a^A(t), P_a] |0\rangle = -F_\phi \langle \phi^a| P_a |0\rangle = \frac{2}{3} \langle \bar{q}q\rangle$$

Here F_ϕ is Goldstone boson (pion) decay constant. Its value is ~ 93 MeV (different normalizations).

- There must exist states for which $\langle 0| A_a^0(0) |n\rangle$ and $\langle 0| P_a |n\rangle$ are non-zero
- It is not vacuum, because $\langle 0| P_a |0\rangle = 0$
- Energy of these states must vanish, because the quark condendate is time independent
- So we need $E_n = 0$
- Such states are massless Goldstone bosons $|\phi^b\rangle$
- GBs are (pseudo)scalars – still to be proven

Dimensions

Field dimensions:

$$\left[\int d^3x\mathcal{L}\right] = [\text{energy}] = 1 \qquad [d^3x] = \left[\text{distance}^3\right] = -3 \;\rightarrow\; [\mathcal{L}] = 4$$

$$4 = [\mathcal{L}_D] = [\bar{q}\partial\!\!\!/ q] = [q]^2 + 1 \rightarrow\; [q] = \frac{3}{2} \;\rightarrow [\langle\bar{q}q\rangle] = 3$$

$$4 = [\mathcal{L}_{YM}] = [F_{\mu\nu}F^{\mu\nu}] = [F_{\mu\nu}]^2 \;\rightarrow\; [F_{\mu\nu}] = 2$$

$$4 = [\mathcal{L}_\phi] = \left[(\partial_\mu\phi)^2\right] \;\rightarrow\; [\phi] = 1$$

Phenomenological values of condensates:

$$\langle\bar{q}q\rangle \simeq -(250\,\text{MeV})^3$$

$$\left\langle\frac{\alpha_s}{\pi}F^a_{\mu\nu}F^{a\,\mu\nu}\right\rangle \simeq (\,400\,\text{GeV})^4$$

Dimension of currents

$$[J_\mu] = [\bar{q}\Gamma_\mu q] = 3$$

Dimensions

In the case of quantum fields there are different conventions. Here we follow:
T-P. Cheng and L-F. Li *Gauge theory of elementary particle physics*

$$\phi_a(\boldsymbol{x},t) \;=\; \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_k}}\left[a_a(\boldsymbol{k})e^{-ik\cdot x}+a_a^\dagger(\boldsymbol{k})e^{+ik\cdot x}\right]$$

$$[\phi_a] \;=\; 1 \to [a_a(\boldsymbol{k})] = -\frac{3}{2}$$

Indeed $\quad \left[a_a(\boldsymbol{k}), a_a^\dagger(\boldsymbol{k}')\right] \;=\; \delta^{(3)}\left(\boldsymbol{k}-\boldsymbol{k}'\right)$

Fock state: $\quad |\phi_a(k)\rangle \;=\; \sqrt{(2\pi)^3 2E_k}\,a_a^\dagger(\boldsymbol{k})\,|0\rangle \to [\,|\phi_a(k)\rangle] = -1$

Matrix element of axial current:

$$\left[\langle 0|\, J_A^{\mu,a}(0)\,\left|\phi^b(p)\right\rangle\right] \;=\; 3-1=2$$
$$\left[ip^\mu F_0\delta^{ab}\right] \;=\; 2$$

Goldstone bosons

We have shown that in QCD axial SU(3) symmetry is spontaneously broken, and this implies the existence of eight Goldstone bosons. What is the effective lagrangian? Natural choice for example:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a^\dagger \partial^\mu \phi_a - V(\phi_a^\dagger \phi_a) \quad \phi_a' = \left[e^{-i\theta_c T^c_{\rm adj}}\right]_{ab} \phi_b = \phi_a - i\theta_c \left(T^c_{\rm adj}\right)_{ab} \phi_b + \dots$$

This lagrangian is invariant under $SU_V(3)$ but it is not clear how it transforms under $SU_A(3)$. We will show, that we can write a lagrangian which is much more "powerfull" (infinte series in powers on field derivatives) and takes explicitly into account $SU_A(3)$ breaking. For this we will need a bit of mathematics.

Consider a hamiltonian $\hat{H}$ (note a "hat"!) which is invariant under a compact Lie group G Moreover, the ground state is invariant only under a subgroup H . We have therefore $n = n_G - n_H$ Goldstone bosons ϕ_i, which are continous, real functions on Minkowski space M^4. Define vector space

$$M_1 \equiv \{\Phi : M^4 \to R^n | \phi_i : M^4 \to R \text{ continuous}\}$$

and find its elements.

based on: Stefan Scherer *Introduction to Chiral Perturbation Theory*, hep-ph/0210398v1

Goldstone bosons

$$M_1 \equiv \{\Phi : M^4 \rightarrow R^n | \phi_i : M^4 \rightarrow R \text{ continuous}\}$$

Define a mapping that associates with each pair $(g, \Phi) \in G \times M_1$

g – group element,

Φ – n component vector with elements ϕ_i

an element $\varphi(g, \Phi) \in M_1$ such that

$$\varphi(e, \Phi) = \Phi \ \forall \ \Phi \in M_1, \ e \text{ identity of } G,$$

$$\varphi(g_1, \varphi(g_2, \Phi)) = \varphi(g_1 g_2, \Phi) \ \forall \ g_1, g_2 \in G, \ \forall \, \Phi \in M_1$$

This is nothing but definition of an operation of G on M_1 . This mapping is not necessarily linear:

$$\varphi(g, \lambda\Phi) \neq \ \lambda\varphi(g, \Phi)$$

Vacuum ("origin" of M_1) $\Phi = 0$ We require that all elements of *G* $h \in H$ map the origin onto itself (little group of $\Phi = 0$)

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Goldstone bosons

- H is not empty, bcause identity maps the origin onto itself
- If $\varphi(h_1, 0) = \varphi(h_2, 0) = 0$ then $\varphi(h_1 h_2, 0) = \varphi(h_1, \varphi(h_2, 0)) = \varphi(h_1, 0) = 0$ which means that $h_1 h_2 \in H$
- Inverse element is also in H: $\varphi(h^{-1}, 0) = \varphi(h^{-1}, \varphi(h, 0)) = \varphi(h^{-1}h, 0) = \varphi(e, 0)$ which means that $h^{-1} \in H$

Define left coset $gH = \{gH | g \in G\}$ (g is fixed) We will establish a connection between the set of all left cosets G/H with the Goldstone boson fileds.

We will check now that all elements of a given coset map the origin onto the same vector in R^n

$$\varphi(gh, 0) = \varphi(g, \varphi(h, 0)) = \varphi(g, 0) \ \forall \ g \in G \text{ and } h \in H$$

These vectors are different if g and g' are "different": $\varphi(g, 0) \neq \varphi(g', 0)$ if $g' \notin gH$
This means that mapping φ is injective with respect to the cosets.

Goldstone bosons

Proof proceeds by negation of the thesis. Assume $\varphi(g,0) = \varphi(g',0)$
Then

$$0 = \varphi(e,0)$$

Goldstone bosons

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However, this implies $g^{-1}g' \in H$ or $g' \in gH$, which contradicts our assumption.

We will now discuss transformations of Φ. To each Φ corresponds a coset $\tilde{g}H$

with $\tilde{g}$ fixed:

$$\Phi = \varphi(f,0) = \varphi(\tilde{g}h,0)$$

Consider transformation of Φ with $\varphi(g)$

$$\varphi(g,\Phi) = \varphi(g,\varphi(\tilde{g}h,0)) = \varphi(g\tilde{g}h,0) = \varphi(f',0) = \Phi' \qquad f' \in g(\tilde{g}H)$$

To obtain transformed Φ' from Φ we need to multiply the left coset $\tilde{g}H$ representing Φ by g to obtain a new left coset represnting Φ'.

Goldstone bosons in QCD

Symmetry group of QCD

$$G = \mathrm{SU}_L(N) \times \mathrm{SU}_R(N) = \{(L,R) | L \in \mathrm{SU}_L(N), R \in \mathrm{SU}_R(N)\}$$

and little group $H = \{(V,V) | V \in \mathrm{SU}(N)\}$ (which is isomorphic to SU(*N*))

Left coset $\tilde{g}H = \{(\tilde{L}V, \tilde{R}V) | V \in \mathrm{SU}(N)\}$ is uniquely characterized by $U = \tilde{R}\tilde{L}^\dagger$

Indeed:

$$(\tilde{L}V, \tilde{R}V) = (\tilde{L}V, \tilde{R}\tilde{L}^\dagger\tilde{L}V) = (1, \tilde{R}\tilde{L}^\dagger)\underbrace{(\tilde{L}V, \tilde{L}V)}_{\in H}, \quad \text{i.e.} \quad \tilde{g}H = (1, \tilde{R}\tilde{L}^\dagger)H$$

(because $\varphi(gh,0) = \varphi(g, \varphi(h,0)) = \varphi(g,0)\ \forall\ g \in G$ and $h \in H$)

Therefore matrix $U = \tilde{R}\tilde{L}^\dagger$ is isomorphic to Φ.

Goldstone bosons in QCD

Now, we will find transformation law for U . Recall $\Phi = \varphi(f,0) = \varphi(\tilde{g}h,0)$

and $\varphi(f',0) = \Phi'$ where $f' = g\tilde{g}h$ or $f' \in g(\tilde{g}H)$. This means, that transformation

o U under $g = (L,R) \in G$ is (recall $\tilde{g}H = (1, \tilde{R}\tilde{L}^\dagger)H$)

$$g\tilde{g}H = (L, R\tilde{R}\tilde{L}^\dagger)H = (1, R\tilde{R}\tilde{L}^\dagger L^\dagger)\underbrace{(L,L)H}_{=H} = (1, R(\tilde{R}\tilde{L}^\dagger)L^\dagger)H$$

Hence we have $U = \tilde{R}\tilde{L}^\dagger \mapsto U' = R(\tilde{R}\tilde{L}^\dagger)L^\dagger = RUL^\dagger$

where we have to reintroduce space-time dependence

$$U(x) \mapsto RU(x)L^\dagger$$

We now see, how the symmetry is broken. Vacuum corresponds to $U \sim 1$ and the symmetry of vacuum is $R = L$.

Nonlinear realization of SU(*N*) x SU(*N*)

We can parametrize SU(*N*) matrix as $U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)$

where for SU(2)

$$\phi(x) = \sum_{i=1}^{3} \tau_i \phi_i(x) = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} \equiv \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$$

or for SU(3)

$$\phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) = \begin{pmatrix} \phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & -\phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix}$$

$$\equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix},$$

[there exist different conventions for signs of particle fields]