QCD lecture 15a

January 18

Current commutators

Full list:

$$
\begin{aligned}
[V_0^a(\vec{x},t), V_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} V_c^\mu(\vec{x},t), \\
[V_0^a(\vec{x},t), V^\mu(\vec{y},t)] &= 0, \\
[V_0^a(\vec{x},t), A_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} A_c^\mu(\vec{x},t), \\
[V_0^a(\vec{x},t), S_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} S_c(\vec{x},t), \\
[V_0^a(\vec{x},t), S_0(\vec{y},t)] &= 0, \\
[V_0^a(\vec{x},t), P_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} P_c(\vec{x},t), \\
[V_0^a(\vec{x},t), P_0(\vec{y},t)] &= 0, \\
[A_0^a(\vec{x},t), V_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} A_c^\mu(\vec{x},t), \\
[A_0^a(\vec{x},t), V^\mu(\vec{y},t)] &= 0, \\
[A_0^a(\vec{x},t), A_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} V_c^\mu(\vec{x},t), \\
[A_0^a(\vec{x},t), S_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} P_c(\vec{x},t), \\
[A_0^a(\vec{x},t), S_0(\vec{y},t)] &= 0, \\
[A_0^a(\vec{x},t), P_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} S_c(\vec{x},t), \\
[A_0^a(\vec{x},t), P_0(\vec{y},t)] &= 0.
\end{aligned}
$$

QCD spectrum

Both vector and axial charges commute with QCD (massless) hamiltonian $H^0_{\rm QCD}$ therefore the eigenstates organize themselves into irreducible representations of the chiral group $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ (axial $\mathrm{U}(1)$ is broken by anomaly). States within each multiplet are (nearly) degenarate in mass and have well defined baryon number ($\mathrm{U}(1)_V$ ensures baryon number conservation). Since axial and vector charges have opposite parity, one would expect that multiplets of opposite parity are degenerate in mass.

For positive parity states: (e.g. baryon or meson ground sates)

$$H^0_{\rm QCD}|i,+\rangle = E_i|i,+\rangle$$

$$P|i,+\rangle = +|i,+\rangle$$

Define now* $|\phi\rangle = Q^a_A|i,+\rangle$ and calculate its mass. Because $[H^0_{\rm QCD}, Q^a_A] = 0$

$$H^0_{\rm QCD}|\phi\rangle = H^0_{\rm QCD}Q^a_A|i,+\rangle = Q^a_A H^0_{\rm QCD}|i,+\rangle = E_i Q^a_A|i,+\rangle = E_i|\phi\rangle$$

so the new state has the same energy (mass) but opposite parity

$$P|\phi\rangle = PQ^a_A P^{-1}P|i,+\rangle = -Q^a_A(+|i,+\rangle) = -|\phi\rangle$$

*charges and generators transforming Hilbert space states are identical (lecture 14)

QCD spectrum

State $|\phi\rangle$ can be expanded in the basis of negative parity multiplet (in fact generators are Clebsch-Gordan coefficients)

$$|\phi\rangle = Q_A^a|i,+\rangle = -t_{ij}^a|j,-\rangle$$

But such degeneracy of opposite parity states is not seen in Nature.

$n^{2s+1}\ell_J$	J^{PC}	I = 1 $u\bar{d}$, $\bar{u}d$, $\frac{1}{\sqrt{2}}(d\bar{d}-u\bar{u})$	I = $\frac{1}{2}$ $u\bar{s}$, $d\bar{s}$; $\bar{d}s$, $\bar{u}s$	I = 0 f'	I = 0 f	θ_{quad} [°]	θ_{lin} [°]
1^1S_0	0^{-+}	$\pi(138)$	$K(494)$	$\eta(548)$	$\eta'(958)$	−11.3	−24.5
1^3S_1	1^{--}	$\rho(770)$	$K^*(892)$	$\phi(1020)$	$\omega(782)$	39.2	36.5
1^1P_1	1^{+-}	$b_1(1235)$	$K_{1B}{}^{\dagger}$	$h_1(1415)$	$h_1(1170)$		
1^3P_0	0^{++}	$a_0(1450)$	$K_0^*(1430)$	$f_0(1710)$	$f_0(1370)$		
1^3P_1	1^{++}	$a_1(1260)$	$K_{1A}{}^{\dagger}$	$f_1(1420)$	$f_1(1285)$		
1^3P_2	2^{++}	$a_2(1320)$	$K_2^*(1430)$	$f_2'(1525)$	$f_2(1270)$	29.6	28.0
1^1D_2	2^{-+}	$\pi_2(1670)$	$K_2(1770)^{\dagger}$	$\eta_2(1870)$	$\eta_2(1645)$		
1^3D_1	1^{--}	$\rho(1700)$	$K^*(1680)^{\ddagger}$		$\omega(1650)$		
1^3D_2	2^{--}		$K_2(1820)^{\dagger}$				
1^3D_3	3^{--}	$\rho_3(1690)$	$K_3^*(1780)$	$\phi_3(1850)$	$\omega_3(1670)$	31.8	30.8
1^3F_4	4^{++}	$a_4(1970)$	$K_4^*(2045)$	$f_4(2300)$	$f_4(2050)$		
1^3G_5	5^{--}	$\rho_5(2350)$	$K_5^*(2380)$				
2^1S_0	0^{-+}	$\pi(1300)$	$K(1460)$	$\eta(1475)$	$\eta(1295)$		
2^3S_1	1^{--}	$\rho(1450)$	$K^*(1410)^{\ddagger}$	$\phi(1680)$	$\omega(1420)$		
2^3P_1	1^{++}	$a_1(1640)$					
2^3P_2	2^{++}	$a_2(1700)$	$K_2^*(1980)$	$f_2(1950)$	$f_2(1640)$		

J^P	(D, L^P_N)	S	Octet members				Singlets
$1/2^+$	$(56,0^+_0)$	1/2	$N(939)$	$\Lambda(1116)$	$\Sigma(1193)$	$\Xi(1318)$	
$1/2^+$	$(56,0^+_2)$	1/2	$N(1440)$	$\Lambda(1600)$	$\Sigma(1660)$	$\Xi(1690)^\dagger$	
$1/2^-$	$(70,1^-_1)$	1/2	$N(1535)$	$\Lambda(1670)$	$\Sigma(1620)$	$\Xi(?)$	$\Lambda(1405)$
					$\Sigma(1560)^\dagger$		
$3/2^-$	$(70,1^-_1)$	1/2	$N(1520)$	$\Lambda(1690)$	$\Sigma(1670)$	$\Xi(1820)$	$\Lambda(1520)$
$1/2^-$	$(70,1^-_1)$	3/2	$N(1650)$	$\Lambda(1800)$	$\Sigma(1750)$	$\Xi(?)$	
					$\Sigma(1620)^\dagger$		
$3/2^-$	$(70,1^-_1)$	3/2	$N(1700)$	$\Lambda(?)$	$\Sigma(1940)^\dagger$	$\Xi(?)$	
$5/2^-$	$(70,1^-_1)$	3/2	$N(1675)$	$\Lambda(1830)$	$\Sigma(1775)$	$\Xi(1950)^\dagger$	
$1/2^+$	$(70,0^+_2)$	1/2	$N(1710)$	$\Lambda(1810)$	$\Sigma(1880)$	$\Xi(?)$	$\Lambda(1810)^\dagger$
$3/2^+$	$(56,2^+_2)$	1/2	$N(1720)$	$\Lambda(1890)$	$\Sigma(?)$	$\Xi(?)$	
$5/2^+$	$(56,2^+_2)$	1/2	$N(1680)$	$\Lambda(1820)$	$\Sigma(1915)$	$\Xi(2030)$	
$7/2^-$	$(70,3^-_3)$	1/2	$N(2190)$	$\Lambda(?)$	$\Sigma(?)$	$\Xi(?)$	$\Lambda(2100)$
$9/2^-$	$(70,3^-_3)$	3/2	$N(2250)$	$\Lambda(?)$	$\Sigma(?)$	$\Xi(?)$	
$9/2^+$	$(56,4^+_4)$	1/2	$N(2220)$	$\Lambda(2350)$	$\Sigma(?)$	$\Xi(?)$	

Spontaneous χSB

What was wrong with the previous argument?
We have tacitly assumed that the ground state of QCD (vacuum) is annihilated by Q_A^a

To show this, consider a creation operator asociated with positive parity fields $a_i^\dagger$ creating positive parity state $|i,+\rangle$ and $b_j^\dagger$ creates quanta of opposite parity. States $|i,+\rangle$ and $|j,-\rangle$ are basis states of an irreducible representation of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$

In analogy with (lecture 14) $[Q^a(t), \Phi_k(\vec{y},t)] = -t^a_{kj}\Phi_j(\vec{y},t)$

we have

$$[Q_A^a, a_i^\dagger] = -t^a_{ij} b_j^\dagger$$

Then

$$Q_A^a|i,+\rangle = Q_A^a a_i^\dagger|0\rangle = \Big([Q_A^a, a_i^\dagger] + a_i^\dagger \underbrace{Q_A^a}_{\hookrightarrow 0}\Big)|0\rangle = -t^a_{ij} b_j^\dagger|0\rangle$$

If axial charges annihilate vacuum then we arrive at

$$|\phi\rangle = Q_A^a|i,+\rangle = -t^a_{ij}|j,-\rangle$$

What happens when $Q_A^a|0\rangle \neq 0$?

Spontaneous χSB

Goldstone theorem:
For each charge (generator) of some symmetry group that does not annihilate vacuum there corresponds a massless particle (Goldstone boson) of parity equal to the parity of this charge. In QCD natural candidates for Goldstone bosons are: π, K and η.

In QCD $Q_V^a|0\rangle = Q_V|0\rangle = 0$ so the vacuum is invariant under $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$
It follows that H^0_{QCD} is also invariant (but not vice versa) and that the physical states correspond to some irreducible representations of $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$

To each $Q_A^a|0\rangle \neq 0$ there corresponds a massless Goldstone boson field $\phi^a(x)$ with zero spin and

$$\phi^a(\vec{x},t) \overset{P}{\mapsto} -\phi^a(-\vec{x},t)$$

Moreover:

$$[Q_V^a, \phi^b(x)] = i f_{abc}\phi^c(x)$$

Quark masses break axial symmetry explicitly, so Goldstone bosons are not exactly massless.

Quark condensate

Recall definitions

$$S_a(y) = \bar{q}(y)\lambda_a q(y), \quad a = 0, \cdots, 8,$$
$$P_a(y) = i\bar{q}(y)\gamma_5\lambda_a q(y), \quad a = 0, \cdots, 8.$$

Generic quark billinears

$$A_i(x) = q^\dagger(x)\hat{A}_i q(x)$$

have the following commutation rules

$$[A_1(\vec{x},t), A_2(\vec{y},t)] = \delta^3(\vec{x}-\vec{y})q^\dagger(x)[\hat{A}_1, \hat{A}_2]q(x)$$

Calculate commutators of vector currents $Q_V^a(t) = \int d^3x q^\dagger(\vec{x},t)\frac{\lambda^a}{2}q(\vec{x},t)$ with S and P

we have $[\frac{\lambda_a}{2}, \gamma_0\lambda_0] = 0$ and $[\frac{\lambda_a}{2}, \gamma_0\lambda_b] = \gamma_0 i f_{abc}\lambda_c$

scalar quark densities transform as a singlet and an octet (similarly pseudoscalars)

$$[Q_V^a(t), S_0(y)] = 0, \quad a = 1, \cdots, 8,$$
$$[Q_V^a(t), S_b(y)] = i\sum_{c=1}^{8} f_{abc}S_c(y), \quad a, b = 1, \cdots, 8$$

Quark condensate

$$[Q_V^a(t), S_b(y)] \ = \ i\sum_{c=1}^{8} f_{abc} S_c(y), \quad a, b = 1, \cdots, 8,$$

One can invert this relation with the help of (recall computation of the Casimir)

$$\sum_{a,b=1}^{8} f_{abc} f_{abd} = 3\delta_{cd}$$

$$S_a(y) = -\frac{i}{3} \sum_{b,c=1}^{8} f_{abc} [Q_V^b(t), S_c(y)]$$

Because vector charges annihilate vacuum $Q_V^a|0\rangle = 0$ we have

$$\langle 0|S_a(y)|0\rangle = \langle 0|S_a(0)|0\rangle \equiv \langle S_a\rangle = 0, \quad a = 1, \cdots, 8$$

where we have used translation invariance of the ground sate:

$$e^{ipy} S(y) e^{-ipy} = S(0)$$

Quark Condensate

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

From $\langle S_a \rangle = 0$ we have:

$a = 3$ $\quad \langle \bar{u}u \rangle - \langle \bar{d}d \rangle = 0$ $\qquad a = 8$ $\quad \langle \bar{u}u \rangle + \langle \bar{d}d \rangle - 2\langle \bar{s}s \rangle = 0$

From these eqs. we have

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle$$

Because $[Q_V^a(t), S_0(y)] = 0, \quad a = 1, \cdots, 8$ the same argument cannot be used for singlet condensate.

However it is clear that

$$0 \neq \langle \bar{q}q \rangle = \langle \bar{u}u + \bar{d}d + \bar{s}s \rangle = 3\langle \bar{u}u \rangle = 3\langle \bar{d}d \rangle = 3\langle \bar{s}s \rangle$$

Quark condensate

Now we shall calculate commutator $i[Q_a^A(t), P_a(y)]$ for fixed a

This is related to

$$(i)^2[\gamma_5 \frac{\lambda_a}{2}, \gamma_0\gamma_5\lambda_a] = \lambda_a^2\gamma_0$$

We have

$$\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4^2 = \lambda_5^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\lambda_6^2 = \lambda_7^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\lambda_8^2 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Quark condensate

Now we shall calculate commutator $i[Q_a^A(t), P_a(y)]$ for fixed a

This is related to

$$(i)^2[\gamma_5 \frac{\lambda_a}{2}, \gamma_0\gamma_5\lambda_a] = \lambda_a^2\gamma_0$$

We have (suppressing y dependence)

$$i[Q_a^A(t), P_a(y)] = \begin{cases} \bar{u}u + \bar{d}d, & a = 1,2,3 \\ \bar{u}u + \bar{s}s, & a = 4,5 \\ \bar{d}d + \bar{s}s, & a = 6,7 \\ \frac{1}{3}(\bar{u}u + \bar{d}d + 4\bar{s}s), & a = 8 \end{cases}$$

which gives vacuum expectation value

$$\langle 0|i[Q_a^A(t), P_a(y)]|0\rangle = \frac{2}{3}\langle \bar{q}q\rangle, \quad a = 1, \cdots, 8$$

Goldstone bosons

Expectation value is non-zero and time independent

$$\langle 0| i[Q_a^A(t), P_a] |0\rangle = i \int d^3x \, \langle 0| \left[A_a^0(x), P_a\right] |0\rangle$$

$$= i \int d^3x \sum\!\!\!\!\!\!\int_n \left\{ \langle 0| A_a^0(x) |n\rangle \langle n| P_a |0\rangle - \langle 0| P_a |n\rangle \langle n| A_a^0(x) |0\rangle \right\}$$

where $$\sum\!\!\!\!\!\!\int_n = \sum_n \int \frac{d^4p_n}{(2\pi)^3} \delta(p_n^2 - m_n^2) = \sum_n \int \frac{d^3p_n}{(2\pi)^3 2p_n^0}$$

$$= i \int d^3x \sum\!\!\!\!\!\!\int_n \left\{ e^{-ip_nx} \langle 0| A_a^0(0) |n\rangle \langle n| P_a |0\rangle - e^{ip_nx} \langle 0| P_a |n\rangle \langle n| A_a^0(0) |0\rangle \right\}$$

only states with zero energy conribute (time indep.) $e^{-ip_nx} = e^{-i(p_n^0 t - \boldsymbol{p}_n \boldsymbol{x})}$

$$\langle 0| i[Q_a^A(t), P_a] |0\rangle = \frac{i}{2} \lim_{p^0 \to 0} \sum_b \int \frac{d^3p}{(2\pi)^3} \int d^3x \left\{ e^{i\boldsymbol{px}} \frac{\langle 0| A_a^0 |\phi^b\rangle}{p^0} \langle \phi^b| P_a |0\rangle - \text{h.c.} \right\}$$

Integral over d^3x gives Dirac delta, which eats up integration over d^3p

Goldstone bosons

$$\langle 0| i[Q_a^A(t), P_a] |0\rangle = \frac{i}{2} \lim_{p^0 \to 0} \sum_b \left\{ \frac{\langle 0| A_a^0 |\phi^b\rangle}{p^0} \langle \phi^b| P_a |0\rangle - \langle 0| P_a |\phi^b\rangle \frac{\langle \phi^b| A_a^0 |0\rangle}{p^0} \right\}$$

From hermicity and Lorentz invariance $\langle 0| A_a^\mu |\phi^b(p)\rangle = i p^\mu F_\phi \delta^{ab}$

and we get

$$\langle 0| i[Q_a^A(t), P_a] |0\rangle = -F_\phi \langle \phi^a| P_a |0\rangle = \frac{2}{3} \langle \bar{q} q\rangle$$

Here F_ϕ is Goldstone boson (pion) decay constant. Its value is ~ 93 MeV (different normalizations).

- There must exist states for which $\langle 0| A_a^0(0) |n\rangle$ and $\langle 0| P_a |n\rangle$ are non-zero
- It is not vacuum, because $\langle 0| P_a |0\rangle = 0$
- Energy of these states must vanish, because the quark condendate is time independent
- So we need $E_n = 0$
- Such states are massless Goldstone bosons $|\phi^b\rangle$
- GBs are (pseudo)scalars – still to be proven