QCD lecture 14b

January 11

Chiral symmetry

Quark masses (from sum rules at μ = 1 GeV)

$$\begin{pmatrix} m_u = 0.005\,\text{GeV} \\ m_d = 0.009\,\text{GeV} \\ m_s = 0.175\,\text{GeV} \end{pmatrix} \ll 1\,\text{GeV} \le \begin{pmatrix} m_c = (1.15 - 1.35)\,\text{GeV} \\ m_b = (4.0 - 4.4)\,\text{GeV} \\ m_t = 174\,\text{GeV} \end{pmatrix}$$

Approximate symmetry: up, down, strange are massless.

QCD lagrangian ($\mathcal{G}_a^{\mu\nu}$ - field tensor)

$$\mathcal{L}^0_{\text{QCD}} = \sum_{l=u,d,s} \bar{q}_l i \not{D}\, q_l - \frac{1}{4}\mathcal{G}_{\mu\nu,a}\mathcal{G}^{\mu\nu}_a$$

We know, that right-handed and left-hanfded fermions transform independently:

$$P_R = \frac{1}{2}(1+\gamma_5) = P_R^\dagger, \quad P_L = \frac{1}{2}(1-\gamma_5) = P_L^\dagger \qquad P_R + P_L = 1,$$

Chiral symmetry

Define

$$q_R = P_R q, \quad q_L = P_L q$$

and rewrite the lagrangian

$$\mathcal{L}^0_{\mathrm{QCD}} = \sum_{l=u,d,s} \left(\bar{q}_{R,l} i \not{D}\, q_{R,l} + \bar{q}_{L,l} i \not{D}\, q_{L,l} \right) - \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}^{\mu\nu}_a$$

Chiral symmetry (global $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$)

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i \sum_{a=1}^{8} \Theta^L_a \frac{\lambda_a}{2} \right) e^{-i\Theta^L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}$$

$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp\left(-i \sum_{a=1}^{8} \Theta^R_a \frac{\lambda_a}{2} \right) e^{-i\Theta^R} \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

Parity

Consider Dirac equation

$$\left\{i\gamma^0\partial_t + i\boldsymbol{\gamma}\cdot\partial_{\mathbf{x}} - m\right\}\psi(t,\mathbf{x}) = 0$$

and space reflection $\mathbf{x} \to -\mathbf{x}$

then
$$\left\{i\gamma^0\partial_t - i\boldsymbol{\gamma}\cdot\partial_{\mathbf{x}} - m\right\}\psi(t,-\mathbf{x}) = 0$$

What is the wave function transformation generated by space reflection?
We have to change sign of space gamma matrices and leave unchanges time gamm matrix:

$$\gamma^0 = P^{-1}\gamma^0 P \qquad\qquad -\boldsymbol{\gamma} = P^{-1}\boldsymbol{\gamma} P$$

Then
$$\left\{i\gamma^0\partial_t + i\boldsymbol{\gamma}\cdot\partial_{\mathbf{x}} - m\right\}\psi^P(t,\mathbf{x}) = 0$$

where

$$\psi^P(t,\mathbf{x}) = P\,\psi(t,-\mathbf{x})$$

Parity

We need to solve $\gamma^0 = P^{-1}\gamma^0 P$ $\quad\quad$ $-\gamma = P^{-1}\gamma P$

and the solution reads (exercise): $P = P^{-1} = \gamma^0$

Parity transformation $\quad P : q(\vec{x}, t) \mapsto \gamma_0 q(-\vec{x}, t)$

changes chirality (because γ^0 anticommutes with γ^5)

$$q_R(\vec{x}, t) = P_R\, q(\vec{x}, t) \mapsto P_R \gamma_0 q(-\vec{x}, t) = \gamma_0 P_L q(-\vec{x}, t) \neq \pm q_R(-\vec{x}, t)$$

Parity transforms left and right fermions into each other.

QCD physical states (mesons) should be grouped in multiplets of some representations of $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$ and, because of the fact that parity transformation changes chiralty, multiplets with positive and negative parity should be gegenrate (in mass). This is not observed experimentally. We will make this statement more precise later.

Conserved currents

Recall Noether theorem:
in order to find conserved currents of some global symmetry transformation, we have to promote this symmetry to a local one and calculate the currents.

Consider $$\mathcal{L} = \mathcal{L}(\Phi_i, \partial_\mu \Phi_i)$$

which leads to the equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} = 0, \quad i = 1, \cdots, n$$

Suppose fields $\Phi_i(x)$ transform according to some symmetry group (local). Consider infinitensimal transformation

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) + \delta\Phi_i(x) = \Phi_i(x) - i\epsilon_a(x) F_i^a[\Phi_j(x)]$$

which is not necessarily linear

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) - i\epsilon_a(x) t_{ij}^a \Phi_j(x)$$

Conserved currents

Field transformation

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) + \delta\Phi_i(x) = \Phi_i(x) - i\epsilon_a(x) F_i^a[\Phi_j(x)]$$

Variation of the lagrangian

$$\begin{aligned}
\delta\mathcal{L} &= \mathcal{L}(\Phi_i', \partial_\mu \Phi_i') - \mathcal{L}(\Phi_i, \partial_\mu \Phi_i) \\
&= \frac{\partial \mathcal{L}}{\partial \Phi_i}\delta\Phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}\partial_\mu \delta\Phi_i \\
&= \epsilon_a(x)\left(-i\frac{\partial \mathcal{L}}{\partial \Phi_i}F_i^a - i\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}\partial_\mu F_i^a\right) + \partial_\mu \epsilon_a(x)\left(-i\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}F_i^a\right)
\end{aligned}$$

Define current

$$J^{\mu,a} = -i\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}F_i^a$$

and calculate
its divergence
and use EoM

$$\begin{aligned}
\partial_\mu J^{\mu,a} &= -i\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}\right)F_i^a - i\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}\partial_\mu F_i^a \\
&= -i\frac{\partial \mathcal{L}}{\partial \Phi_i}F_i^a - i\frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i}\partial_\mu F_i^a,
\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} = 0$$

Conserved currents

We arrive at $\delta\mathcal{L} = \epsilon_a(x)\partial_\mu J^{\mu,a} + \partial_\mu\epsilon_a(x)J^{\mu,a}$

This allows do define currents and current derivarives as

$$\begin{aligned} J^{\mu,a} &= \frac{\partial\delta\mathcal{L}}{\partial\partial_\mu\epsilon_a}, \\ \partial_\mu J^{\mu,a} &= \frac{\partial\delta\mathcal{L}}{\partial\epsilon_a}. \end{aligned}$$

If we demand the action to be invariant under global transformation, we conclude that the current is conserved:

$$\partial_\mu J^{\mu,a} = 0$$

It follows that there exists a conserved charge (exercise)

$$Q^a(t) = \int d^3x J_0^a(\vec{x}, t)$$

Currents in QFT

Canonical quantization
define generalized momenta $\Pi_i = \partial\mathcal{L}/\partial(\partial_0\Phi_i)$

and impose commutation rules:

$$\begin{aligned} [\Phi_i(\vec{x},t), \Pi_j(\vec{y},t)] &= i\delta^3(\vec{x}-\vec{y})\delta_{ij}, \\ [\Phi_i(\vec{x},t), \Phi_j(\vec{y},t)] &= 0, \\ [\Pi_i(\vec{x},t), \Pi_j(\vec{y},t)] &= 0. \end{aligned}$$

Suppose now that the symmetry transformation is linear

$$\Phi_i(x) \mapsto \Phi'_i(x) = \Phi_i(x) - i\epsilon_a(x) t^a_{ij} \Phi_j(x)$$

then (current and charge are operators now, normal ordering suppressed)

$$\begin{aligned} J^{\mu,a}(x) &= -it^a_{ij} \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\Phi_j, \\ Q^a(t) &= -i\int d^3x\, \Pi_i(x) t^a_{ij} \Phi_j(x) \end{aligned}$$

Commutation rules

It is easy to show (exercise)

$$
\begin{aligned}
[Q^a(t), \Phi_k(\vec{y}, t)] &= -it^a_{ij} \int d^3x \, [\Pi_i(\vec{x}, t)\Phi_j(\vec{x}, t), \Phi_k(\vec{y}, t)] \\
&= -t^a_{kj} \Phi_j(\vec{y}, t).
\end{aligned}
$$

Field (operator) transformations induce transformations of the Hilbert space

$$|\alpha'\rangle = [1 + i\epsilon_a G^a(t)]|\alpha\rangle$$

where G^a are hermitian operators (they in principle could depend on time).
We demand

$$\langle \beta | A | \alpha \rangle = \langle \beta' | A' | \alpha' \rangle$$

Commutation rules

For a matrix element of a filed we have

$$\begin{aligned}\langle\beta|\Phi_i(x)|\alpha\rangle &= \langle\beta'|\Phi_i'(x)|\alpha'\rangle \\ &= \langle\beta|[1-i\epsilon_a G^a(t)][\Phi_i(x)-i\epsilon_b t^b_{ij}\Phi_j(x)][1+i\epsilon_c G^c(t)]|\alpha\rangle\end{aligned}$$

terms linear in ε should vanish

$$0=-i\epsilon_a[G^a(t),\Phi_i(x)] \underbrace{-i\epsilon_a t^a_{ij}\Phi_j(x)}_{i\epsilon_a[Q^a(t),\Phi_i(x)]} \;,$$

From this we conclude that $G^a(t)=Q^a(t)$

Commutation rules

Finally

$$[Q^a(t), Q^b(t)] \;=\; -i(t^a_{ij}t^b_{jk} - t^b_{ij}t^a_{jk}) \int d^3x\, \Pi_i(\vec{x},t)\Phi_k(\vec{x},t)$$

Recalling that

$$t^a_{ij}t^b_{jk} - t^b_{ij}t^a_{jk} = iC_{abc}t^c_{ik}$$

We have

$$[Q^a(t), Q^b(t)] = iC_{abc}Q^c(t)$$

QCD currents

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i\sum_{a=1}^{8}\Theta_a^L\frac{\lambda_a}{2}\right)e^{-i\Theta^L}\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}$$

$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp\left(-i\sum_{a=1}^{8}\Theta_a^R\frac{\lambda_a}{2}\right)e^{-i\Theta^R}\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

Repeating the same steps we arrive at (for massless fermions)

$$\delta\mathcal{L}^0_{\mathrm{QCD}} = \bar{q}_R\left(\sum_{a=1}^{8}\partial_\mu\Theta_a^R\frac{\lambda_a}{2} + \partial_\mu\Theta^R\right)\gamma^\mu q_R + \bar{q}_L\left(\sum_{a=1}^{8}\partial_\mu\Theta_a^L\frac{\lambda_a}{2} + \partial_\mu\Theta^L\right)\gamma^\mu q_L$$

and (quark fileds are now operators) we have 18 conserved currents:

$$\begin{aligned} L^{\mu,a} &= \bar{q}_L\gamma^\mu\frac{\lambda^a}{2}q_L, \quad \partial_\mu L^{\mu,a} = 0, \\ R^{\mu,a} &= \bar{q}_R\gamma^\mu\frac{\lambda^a}{2}q_R, \quad \partial_\mu R^{\mu,a} = 0. \end{aligned}$$

QCD currents

Define vector and axial currents

octet vector
$$V^{\mu,a} = R^{\mu,a} + L^{\mu,a} = \bar{q}\gamma^{\mu}\frac{\lambda^a}{2}q,$$

axial (exercise)
$$A^{\mu,a} = R^{\mu,a} - L^{\mu,a} = \bar{q}\gamma^{\mu}\gamma_5\frac{\lambda^a}{2}q,$$

singlet vector
$$V^{\mu} = \bar{q}_R\gamma^{\mu}q_R + \bar{q}_L\gamma^{\mu}q_L = \bar{q}\gamma^{\mu}q$$

axial (exercise)
$$A^{\mu} = \bar{q}_R\gamma^{\mu}q_R - \bar{q}_L\gamma^{\mu}q_L = \bar{q}\gamma^{\mu}\gamma_5 q$$

All these currents are conserved (modulo anomaly)

Parity of currents

Parity operator: γ^0

Transformation properties of gamma matrices

Γ	1	γ^μ	$\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$
$\gamma_0\Gamma\gamma_0$	1	γ_μ	$\sigma_{\mu\nu}$	$-\gamma_5$	$-\gamma_\mu\gamma_5$

imply the following properties of currents

$$P : V^{\mu,a}(\vec{x},t) \mapsto V^a_\mu(-\vec{x},t),$$
$$P : A^{\mu,a}(\vec{x},t) \mapsto -A^a_\mu(-\vec{x},t).$$

QCD charges

$$
\begin{aligned}
Q_L^a(t) &= \int d^3x\, q_L^\dagger(\vec{x},t)\frac{\lambda^a}{2}q_L(\vec{x},t), \quad a=1,\cdots,8, \\
Q_R^a(t) &= \int d^3x\, q_R^\dagger(\vec{x},t)\frac{\lambda^a}{2}q_R(\vec{x},t), \quad a=1,\cdots,8, \\
Q_V(t) &= \int d^3x \left[q_L^\dagger(\vec{x},t)q_L(\vec{x},t)+q_R^\dagger(\vec{x},t)q_R(\vec{x},t)\right].
\end{aligned}
$$

Recall anti-commutation relations for quark fields

$$
\begin{aligned}
\{q_{\alpha,r}(\vec{x},t), q_{\beta,s}^\dagger(\vec{y},t)\} &= \delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta}\delta_{rs} \\
\{q_{\alpha,r}(\vec{x},t), q_{\beta,s}(\vec{y},t)\} &= 0, \\
\{q_{\alpha,r}^\dagger(\vec{x},t), q_{\beta,s}^\dagger(\vec{y},t)\} &= 0,
\end{aligned}
$$

Commutators

To compute current commutators that are billinears in quark fields, we will use

$$\left[q^\dagger(\boldsymbol{x},t)\Gamma^{(1)}T^{(1)}q(\boldsymbol{x},t),\, q^\dagger(\boldsymbol{y},t)\Gamma^{(2)}T^{(2)}q(\boldsymbol{y},t)\right]$$

$$= \Gamma^{(1)}_{\alpha\beta}\Gamma^{(2)}_{\sigma\tau}\, T^{(1)}_{pq}T^{(2)}_{rs}\left[q^\dagger_{\alpha p}(\boldsymbol{x},t)q_{\beta q}(\boldsymbol{x},t),\, q^\dagger_{\sigma r}(\boldsymbol{y},t)q_{\tau s}(\boldsymbol{y},t)\right]$$

the identity

$$[ab, cd] = a\{b,c\}d - ac\{b,d\} + \{a,c\}db - c\{a,d\}b.$$

and cannonical anti-commutation rules

$$\begin{aligned}
\{q_{\alpha,r}(\vec{x},t), q^\dagger_{\beta,s}(\vec{y},t)\} &= \delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta}\delta_{rs}, \\
\{q_{\alpha,r}(\vec{x},t), q_{\beta,s}(\vec{y},t)\} &= 0, \\
\{q^\dagger_{\alpha,r}(\vec{x},t), q^\dagger_{\beta,s}(\vec{y},t)\} &= 0,
\end{aligned}$$

QCD commutation rules

QCD charges form Lie algebra (exercise)

$$
\begin{aligned}
[Q_L^a, Q_L^b] &= i f_{abc} Q_L^c, \\
[Q_R^a, Q_R^b] &= i f_{abc} Q_R^c, \\
[Q_L^a, Q_R^b] &= 0, \\
[Q_L^a, Q_V] &= [Q_R^a, Q_V] = 0
\end{aligned}
$$

of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ group

For conserved charges:

$$[Q_L^a, H_{\mathrm{QCD}}^0] = [Q_R^a, H_{\mathrm{QCD}}^0] = [Q_V, H_{\mathrm{QCD}}^0] = 0$$

Axial current is anomalous, but otherwise it would commute with the hamiltonian as well.

QCD commutation rules

QCD charges form Lie algebra (exercise)

$$
\begin{aligned}
[Q_L^a, Q_L^b] &= i f_{abc} Q_L^c, \\
[Q_R^a, Q_R^b] &= i f_{abc} Q_R^c, \\
[Q_L^a, Q_R^b] &= 0, \\
[Q_L^a, Q_V] &= [Q_R^a, Q_V] = 0
\end{aligned}
$$

$$
\begin{aligned}
[Q_V^a, Q_V^b] &= i f_{abc} Q_V^c \\
[Q_A^a, Q_A^b] &= i f_{abc} Q_V^c \\
[Q_V^a, Q_A^b] &= i f_{abc} Q_A^c
\end{aligned}
$$

of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ group

For conserved charges:

$$[Q_L^a, H^0_{\mathrm{QCD}}] = [Q_R^a, H^0_{\mathrm{QCD}}] = [Q_V, H^0_{\mathrm{QCD}}] = 0$$

Axial current is anomalous, but otherwise it would commute with the hamiltonian as well.

Quark masses – χSB

(chiral symmetry breaking)

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$$= \frac{m_u + m_d + m_s}{\sqrt{6}}\lambda_0 + \frac{(m_u + m_d)/2 - m_s}{\sqrt{3}}\lambda_8 + \frac{m_u - m_d}{2}\lambda_3.$$

$$\lambda_0 = \sqrt{\frac{2}{3}}\mathbf{1}$$

Symmetry breaking lagrangian:

$$\mathcal{L}_M = -\bar{q}Mq = -(\bar{q}_R M q_L + \bar{q}_L M q_R)$$

Now we calculate variation of $\mathcal{L}_M$ under chiral transformations

$$\exp\left(-i\sum_{a=1}^{8}\Theta_a^L\frac{\lambda_a}{2}\right)e^{-i\Theta^L} \quad \text{and} \quad \exp\left(-i\sum_{a=1}^{8}\Theta_a^R\frac{\lambda_a}{2}\right)e^{-i\Theta^R}$$

Quark masses – χSB

(chiral symmetry breaking)

$$\delta\mathcal{L}_M = -i\left[\sum_{a=1}^{8}\Theta_a^R\left(\bar{q}_R\frac{\lambda_a}{2}Mq_L - \bar{q}_LM\frac{\lambda_a}{2}q_R\right) + \Theta^R\left(\bar{q}_RMq_L - \bar{q}_LMq_R\right)\right.$$
$$\left. + \sum_{a=1}^{8}\Theta_a^L\left(\bar{q}_L\frac{\lambda_a}{2}Mq_R - \bar{q}_RM\frac{\lambda_a}{2}q_L\right) + \Theta^L\left(\bar{q}_LMq_R - \bar{q}_RMq_L\right)\right],$$

From this we can easily calculate currents and current derivatives (lecture 9):

$$\partial_\mu L^{\mu,a} = \frac{\partial\delta\mathcal{L}_M}{\partial\Theta_a^L} = -i\left(\bar{q}_L\frac{\lambda_a}{2}Mq_R - \bar{q}_RM\frac{\lambda_a}{2}q_L\right),$$
$$\partial_\mu R^{\mu,a} = \frac{\partial\delta\mathcal{L}_M}{\partial\Theta_a^R} = -i\left(\bar{q}_R\frac{\lambda_a}{2}Mq_L - \bar{q}_LM\frac{\lambda_a}{2}q_R\right),$$
$$\partial_\mu L^{\mu} = \frac{\partial\delta\mathcal{L}_M}{\partial\Theta^L} = -i\left(\bar{q}_LMq_R - \bar{q}_RMq_L\right),$$
$$\partial_\mu R^{\mu} = \frac{\partial\delta\mathcal{L}_M}{\partial\Theta^R} = -i\left(\bar{q}_RMq_L - \bar{q}_LMq_R\right).$$

Quark masses – χSB

(chiral symmetry breaking)

$$\partial_\mu V^{\mu,a} = i\bar{q}[M, \frac{\lambda_a}{2}]q,$$

$$\partial_\mu A^{\mu,a} = i\left(\bar{q}_L\{\frac{\lambda_a}{2}, M\}q_R - \bar{q}_R\{\frac{\lambda_a}{2}, M\}q_L\right) = i\bar{q}\{\frac{\lambda_a}{2}, M\}\gamma_5 q,$$

$$\partial_\mu V^\mu = 0,$$

$$\partial_\mu A^\mu = 2i\bar{q}M\gamma_5 q + \text{ anomaly}$$

Here we included anomaly, but for most of the time we will ignore it.

- Individual vector currents $\bar{u}\gamma^\mu u, \bar{d}\gamma^\mu d$ and $\bar{s}\gamma^\mu s$ are always conserved
- Vector current is a sum of them and is also conserved
- Baryon number is conserved
- Axial current is not conserved due to the quark masses (and anomaly)
- For equal quark mass all vector currents $V^{\mu,a}$ are conserved
- Axial flavor currents $A^{\mu,a}$ are not conserved, but their divergences are propotional to pseudoscalar densities. This leads to the concept of partially conserved axial currents (PCAC).

Conserved currents

We arrive at $\delta\mathcal{L} = \epsilon_a(x)\partial_\mu J^{\mu,a} + \partial_\mu\epsilon_a(x) J^{\mu,a}$

This allows do define currents and current derivarives as

$$J^{\mu,a} = \frac{\partial\delta\mathcal{L}}{\partial\partial_\mu\epsilon_a},$$
$$\partial_\mu J^{\mu,a} = \frac{\partial\delta\mathcal{L}}{\partial\epsilon_a}.$$

If we demand the action to be invariant under global transformation, we conclude that the current is conserved:

$$\partial_\mu J^{\mu,a} = 0$$

It follows that there exists a consrved charge (exercise)

$$Q^a(t) = \int d^3x J_0^a(\vec{x}, t)$$

QCD currents

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i\sum_{a=1}^{8}\Theta_a^L\frac{\lambda_a}{2}\right)e^{-i\Theta^L}\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}$$

$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp\left(-i\sum_{a=1}^{8}\Theta_a^R\frac{\lambda_a}{2}\right)e^{-i\Theta^R}\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

Repeating the same steps we arrive at (for massless fermions)

$$\delta\mathcal{L}^0_{\mathrm{QCD}} = \bar{q}_R\left(\sum_{a=1}^{8}\partial_\mu\Theta_a^R\frac{\lambda_a}{2} + \partial_\mu\Theta^R\right)\gamma^\mu q_R + \bar{q}_L\left(\sum_{a=1}^{8}\partial_\mu\Theta_a^L\frac{\lambda_a}{2} + \partial_\mu\Theta^L\right)\gamma^\mu q_L$$

and (quark fileds are now operators) we have 18 conserved currents:

$$\begin{aligned} L^{\mu,a} &= \bar{q}_L\gamma^\mu\frac{\lambda^a}{2}q_L, \quad \partial_\mu L^{\mu,a} = 0, \\ R^{\mu,a} &= \bar{q}_R\gamma^\mu\frac{\lambda^a}{2}q_R, \quad \partial_\mu R^{\mu,a} = 0. \end{aligned}$$

QCD currents

Define vector and axial currents

octet vector

$$V^{\mu,a} = R^{\mu,a} + L^{\mu,a} = \bar{q}\gamma^{\mu}\frac{\lambda^a}{2}q,$$

axial (exercise)

$$A^{\mu,a} = R^{\mu,a} - L^{\mu,a} = \bar{q}\gamma^{\mu}\gamma_5\frac{\lambda^a}{2}q,$$

singlet vector

$$V^{\mu} = \bar{q}_R\gamma^{\mu}q_R + \bar{q}_L\gamma^{\mu}q_L = \bar{q}\gamma^{\mu}q$$

axial (exercise)

$$A^{\mu} = \bar{q}_R\gamma^{\mu}q_R - \bar{q}_L\gamma^{\mu}q_L = \bar{q}\gamma^{\mu}\gamma_5 q$$

All these currents are conserved (modulo anomaly)

Parity of currents

Parity operator: γ^0

Transformation properties of gamma matrices

Γ	1	γ^μ	$\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$
$\gamma_0\Gamma\gamma_0$	1	γ_μ	$\sigma_{\mu\nu}$	$-\gamma_5$	$-\gamma_\mu\gamma_5$

imply the following properties of currents

$$P : V^{\mu,a}(\vec{x},t) \mapsto V_\mu^a(-\vec{x},t),$$
$$P : A^{\mu,a}(\vec{x},t) \mapsto -A_\mu^a(-\vec{x},t).$$

QCD charges

$$
\begin{aligned}
Q_L^a(t) &= \int d^3x \, q_L^\dagger(\vec{x},t)\frac{\lambda^a}{2}q_L(\vec{x},t), \quad a=1,\cdots,8, \\
Q_R^a(t) &= \int d^3x \, q_R^\dagger(\vec{x},t)\frac{\lambda^a}{2}q_R(\vec{x},t), \quad a=1,\cdots,8, \\
Q_V(t) &= \int d^3x \left[q_L^\dagger(\vec{x},t)q_L(\vec{x},t)+q_R^\dagger(\vec{x},t)q_R(\vec{x},t)\right].
\end{aligned}
$$

Recall anti-commutation relations for quark fields

$$
\begin{aligned}
\{q_{\alpha,r}(\vec{x},t), q_{\beta,s}^\dagger(\vec{y},t)\} &= \delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta}\delta_{rs} \\
\{q_{\alpha,r}(\vec{x},t), q_{\beta,s}(\vec{y},t)\} &= 0, \\
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\end{aligned}
$$

Commutators

To compute current commutators that are billinears in quark fields, we will use

$$\left[q^\dagger(\boldsymbol{x},t)\Gamma^{(1)}T^{(1)}q(\boldsymbol{x},t),\,q^\dagger(\boldsymbol{y},t)\Gamma^{(2)}T^{(2)}q(\boldsymbol{y},t)\right]$$

$$= \Gamma^{(1)}_{\alpha\beta}\Gamma^{(2)}_{\sigma\tau}\,T^{(1)}_{pq}T^{(2)}_{rs}\left[q^\dagger_{\alpha p}(\boldsymbol{x},t)q_{\beta q}(\boldsymbol{x},t),\,q^\dagger_{\sigma r}(\boldsymbol{y},t)q_{\tau s}(\boldsymbol{y},t)\right]$$

the identity

$$[ab,cd] = a\{b,c\}d - ac\{b,d\} + \{a,c\}db - c\{a,d\}b.$$

and cannonical anti-commutation rules

$$\begin{aligned}
\{q_{\alpha,r}(\vec{x},t),\,q^\dagger_{\beta,s}(\vec{y},t)\} &= \delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta}\delta_{rs},\\
\{q_{\alpha,r}(\vec{x},t),\,q_{\beta,s}(\vec{y},t)\} &= 0,\\
\{q^\dagger_{\alpha,r}(\vec{x},t),\,q^\dagger_{\beta,s}(\vec{y},t)\} &= 0,
\end{aligned}$$

QCD commutation rules

QCD charges form Lie algebra (exercise)

$$
\begin{aligned}
[Q_L^a, Q_L^b] &= i f_{abc} Q_L^c, \\
[Q_R^a, Q_R^b] &= i f_{abc} Q_R^c, \\
[Q_L^a, Q_R^b] &= 0, \\
[Q_L^a, Q_V] &= [Q_R^a, Q_V] = 0
\end{aligned}
$$

of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ group

For conserved charges:

$$[Q_L^a, H_{\mathrm{QCD}}^0] = [Q_R^a, H_{\mathrm{QCD}}^0] = [Q_V, H_{\mathrm{QCD}}^0] = 0$$

Axial current is anomalous, but otherwise it would commute with the hamiltonian as well.

QCD commutation rules

QCD charges form Lie algebra (exercise)

$$
\begin{aligned}
[Q_L^a, Q_L^b] &= i f_{abc} Q_L^c, \\
[Q_R^a, Q_R^b] &= i f_{abc} Q_R^c, \\
[Q_L^a, Q_R^b] &= 0, \\
[Q_L^a, Q_V] &= [Q_R^a, Q_V] = 0
\end{aligned}
$$

$$
\begin{aligned}
[Q_V^a, Q_V^b] &= i f_{abc} Q_V^c \\
[Q_A^a, Q_A^b] &= i f_{abc} Q_V^c \\
[Q_V^a, Q_A^b] &= i f_{abc} Q_A^c
\end{aligned}
$$

of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ group

For conserved charges:

$$[Q_L^a, H^0_{\mathrm{QCD}}] = [Q_R^a, H^0_{\mathrm{QCD}}] = [Q_V, H^0_{\mathrm{QCD}}] = 0$$

Axial current is anomalous, but otherwise it would commute with the hamiltonian as well.

Quark masses – χSB

(chiral symmetry breaking)

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

$$= \frac{m_u + m_d + m_s}{\sqrt{6}}\lambda_0 + \frac{(m_u + m_d)/2 - m_s}{\sqrt{3}}\lambda_8 + \frac{m_u - m_d}{2}\lambda_3.$$

$$\lambda_0 = \sqrt{\frac{2}{3}}\mathbf{1}$$

Symmetry breaking lagrangian:

$$\mathcal{L}_M = -\bar{q}Mq = -(\bar{q}_R M q_L + \bar{q}_L M q_R)$$

Now we calculate variation of $\mathcal{L}_M$ under chiral transformations

$$\exp\left(-i\sum_{a=1}^{8}\Theta_a^L \frac{\lambda_a}{2}\right) e^{-i\Theta^L} \quad \text{and} \quad \exp\left(-i\sum_{a=1}^{8}\Theta_a^R \frac{\lambda_a}{2}\right) e^{-i\Theta^R}$$

Quark masses – χSB

(chiral symmetry breaking)

$$\begin{aligned}\delta\mathcal{L}_M &= -i\Bigg[\sum_{a=1}^{8}\Theta_a^R\left(\bar{q}_R\frac{\lambda_a}{2}Mq_L-\bar{q}_LM\frac{\lambda_a}{2}q_R\right)+\Theta^R\left(\bar{q}_RMq_L-\bar{q}_LMq_R\right)\\&+\sum_{a=1}^{8}\Theta_a^L\left(\bar{q}_L\frac{\lambda_a}{2}Mq_R-\bar{q}_RM\frac{\lambda_a}{2}q_L\right)+\Theta^L\left(\bar{q}_LMq_R-\bar{q}_RMq_L\right)\Bigg],\end{aligned}$$

From this we can easily calculate currents and current derivatives (lecture 9):

$$\begin{aligned}\partial_\mu L^{\mu,a} &= \frac{\partial\delta\mathcal{L}_M}{\partial\Theta_a^L}=-i\left(\bar{q}_L\frac{\lambda_a}{2}Mq_R-\bar{q}_RM\frac{\lambda_a}{2}q_L\right),\\ \partial_\mu R^{\mu,a} &= \frac{\partial\delta\mathcal{L}_M}{\partial\Theta_a^R}=-i\left(\bar{q}_R\frac{\lambda_a}{2}Mq_L-\bar{q}_LM\frac{\lambda_a}{2}q_R\right),\\ \partial_\mu L^{\mu} &= \frac{\partial\delta\mathcal{L}_M}{\partial\Theta^L}=-i\left(\bar{q}_LMq_R-\bar{q}_RMq_L\right),\\ \partial_\mu R^{\mu} &= \frac{\partial\delta\mathcal{L}_M}{\partial\Theta^R}=-i\left(\bar{q}_RMq_L-\bar{q}_LMq_R\right).\end{aligned}$$

Quark masses – χSB

(chiral symmetry breaking)

$$\partial_\mu V^{\mu,a} = i\bar{q}[M, \frac{\lambda_a}{2}]q,$$

$$\partial_\mu A^{\mu,a} = i\left(\bar{q}_L\{\frac{\lambda_a}{2}, M\}q_R - \bar{q}_R\{\frac{\lambda_a}{2}, M\}q_L\right) = i\bar{q}\{\frac{\lambda_a}{2}, M\}\gamma_5 q,$$

$$\partial_\mu V^\mu = 0,$$

$$\partial_\mu A^\mu = 2i\bar{q}M\gamma_5 q + \text{ anomaly}$$

Here we included anomaly, but for most of the time we will ignore it.

- Individual vector currents $\bar{u}\gamma^\mu u, \bar{d}\gamma^\mu d$ and $\bar{s}\gamma^\mu s$ are always conserved
- Vector current is a sum of them and is also conserved
- Baryon number is conserved
- Axial current is not conserved due to the quark masses (and anomaly)
- For equal quark mass all vector currents $V^{\mu,a}$ are conserved
- Axial flavor currents $A^{\mu,a}$ are not conserved, but their divergences are propotional to pseudoscalar densities. This leads to the concept of partially conserved axial currents (PCAC).

Chiral Ward identities

Define densities:

$$S_a(x) = \bar{q}(x)\lambda_a q(x), \quad P_a(x) = i\bar{q}(x)\gamma_5\lambda_a q(x), \quad a = 0, \cdots, 8$$

$$S(x) = \bar{q}(x)q(x), \quad P(x) = i\bar{q}(x)\gamma_5 q(x)$$

Ward identities relate divrgences of Green functions containing at least one current $V^{\mu,a}$ or $A^{\mu,a}$ to some linear combinations of other Green functions.

Example:

$$\begin{aligned} G^{\mu,ab}_{AP}(x,y) &= \langle 0|T[A^\mu_a(x)P_b(y)]|0\rangle \\ &= \Theta(x_0 - y_0)\langle 0|A^\mu_a(x)P_b(y)|0\rangle + \Theta(y_0 - x_0)\langle 0|P_b(y)A^\mu_a(x)|0\rangle \end{aligned}$$

We shall calculate: $\partial^x_\mu G^{\mu,ab}_{AP}(x,y)$ remembering that

$$\partial^x_\mu \Theta(x_0 - y_0) = \delta(x_0 - y_0)g_{0\mu} = -\partial^x_\mu \Theta(y_0 - x_0)$$

Chiral Ward identities

Differentiating

$$
\begin{aligned}
G_{AP}^{\mu,ab}(x,y) &= \langle 0|T[A_a^\mu(x)P_b(y)]|0\rangle \\
&= \Theta(x_0-y_0)\langle 0|A_a^\mu(x)P_b(y)|0\rangle + \Theta(y_0-x_0)\langle 0|P_b(y)A_a^\mu(x)|0\rangle
\end{aligned}
$$

we get:

$$
\begin{aligned}
&\partial_\mu^x G_{AP}^{\mu,ab}(x,y) \\
&= \delta(x_0-y_0)\langle 0|A_0^a(x)P_b(y)|0\rangle - \delta(x_0-y_0)\langle 0|P_b(y)A_0^a(x)|0\rangle \\
&\quad +\Theta(x_0-y_0)\langle 0|\partial_\mu^x A_a^\mu(x)P_b(y)|0\rangle + \Theta(y_0-x_0)\langle 0|P_b(y)\partial_\mu^x A_a^\mu(x)|0\rangle \\
&= \delta(x_0-y_0)\langle 0|[A_0^a(x),P_b(y)]|0\rangle + \langle 0|T[\partial_\mu^x A_a^\mu(x)P_b(y)]|0\rangle,
\end{aligned}
$$

equal time commutator can be calculated from chiral algebra.

time ordered product for conserved current this term is zero

Chiral Ward identities

Generalization

$$\partial_\mu^x \langle 0|T\{J^\mu(x)A_1(x_1)\cdots A_n(x_n)\}|0\rangle =$$

$$\begin{aligned} &= \langle 0|T\{[\partial_\mu^x J^\mu(x)]A_1(x_1)\cdots A_n(x_n)\}|0\rangle \\ &\quad +\delta(x^0-x_1^0)\langle 0|T\{[J_0(x),A_1(x_1)]A_2(x_2)\cdots A_n(x_n)\}|0\rangle \\ &\quad +\delta(x^0-x_2^0)\langle 0|T\{A_1(x_1)[J_0(x),A_2(x_2)]\cdots A_n(x_n)\}|0\rangle \\ &\quad +\cdots+\delta(x^0-x_n^0)\langle 0|T\{A_1(x_1)\cdots[J_0(x),A_n(x_n)]\}|0\rangle \end{aligned}$$

Current commutators

Full list:

$$
\begin{aligned}
[V_0^a(\vec{x},t), V_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} V_c^\mu(\vec{x},t), \\
[V_0^a(\vec{x},t), V^\mu(\vec{y},t)] &= 0, \\
[V_0^a(\vec{x},t), A_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} A_c^\mu(\vec{x},t), \\
[V_0^a(\vec{x},t), S_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} S_c(\vec{x},t), \\
[V_0^a(\vec{x},t), S_0(\vec{y},t)] &= 0, \\
[V_0^a(\vec{x},t), P_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} P_c(\vec{x},t), \\
[V_0^a(\vec{x},t), P_0(\vec{y},t)] &= 0, \\
[A_0^a(\vec{x},t), V_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} A_c^\mu(\vec{x},t), \\
[A_0^a(\vec{x},t), V^\mu(\vec{y},t)] &= 0, \\
[A_0^a(\vec{x},t), A_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} V_c^\mu(\vec{x},t), \\
[A_0^a(\vec{x},t), S_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} P_c(\vec{x},t), \\
[A_0^a(\vec{x},t), S_0(\vec{y},t)] &= 0, \\
[A_0^a(\vec{x},t), P_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} S_c(\vec{x},t), \\
[A_0^a(\vec{x},t), P_0(\vec{y},t)] &= 0.
\end{aligned}
$$

Schwinger terms*

Schwinger has shown that naive commutation rules involving charge densities have extra contributions:

$$[J_0^a(\vec{x},0), J_i^b(\vec{y},0)] = iC_{abc}J_i^c(\vec{x},0)\delta^3(\vec{x}-\vec{y}) + S_{ij}^{ab}(\vec{y},0)\partial^j\delta^3(\vec{x}-\vec{y}),$$

where the Schwinger term satisfies

$$S_{ij}^{ab}(\vec{y},0) = S_{ji}^{ba}(\vec{y},0)$$

One can get rid of the Schwinger terms by redefining the time ordered product. In what follows we shall ignore Schwinger terms.

S. Treiman, R. Jackiw, and D. J. Gross, *Lectures on Current Algebra and Its Applications* (Princeton University Press, Princeton, 1972).

Chiral Ward identities

Example: $G_{AP}^{\mu,ab}(x,y) = \langle 0|T[A_a^\mu(x)P_b(y)]|0\rangle$

we have shown: $\partial_\mu^x G_{AP}^{\mu,ab}(x,y)$

$$= \delta(x_0-y_0)\langle 0|[A_0^a(x),P_b(y)]|0\rangle + \langle 0|T[\partial_\mu^x A_a^\mu(x)P_b(y)]|0\rangle$$

$$= \delta^4(x-y)if_{abc}\langle 0|S_c(x)|0\rangle$$ < follows from symmetry

symmetry breaking > $+i\langle 0|T[\bar{q}(x)\{\frac{\lambda_a}{2},M\}\gamma_5 q(x)P_b(y)]|0\rangle$

We can now calculate the anti-commutator (no summation over a) [exercise]

$$i\bar{q}(x)\{\frac{\lambda_a}{2},M\}\gamma_5 q(x) =$$

$$\left[\frac{1}{3}(m_u+m_d+m_s)+\frac{1}{\sqrt{3}}\left(\frac{m_u+m_d}{2}-m_s\right)d_{aa8}\right]P_a(x)$$

$$+\left[\sqrt{\frac{1}{6}}(m_u-m_d)\delta_{a3}+\frac{\sqrt{2}}{3}\left(\frac{m_u+m_d}{2}-m_s\right)\delta_{a8}\right]P_0(x)$$

$$+\frac{m_u-m_d}{2}\sum_{c=1}^{8}d_{a3c}P_c(x).$$

Chiral Ward identities

Another example (for SU(2) and for $m_u = m_d = m$):

$$\partial^\mu A^i_\mu \;\; = \;\; im \left(\bar{q}\tau^i\gamma_5 q\right)$$

Consider nucleon matrix element

$$\langle N(p_f)|A^i_\mu(x)|N(p_i)\rangle = \langle N(p_f)|\bar{q}(x)\gamma_\mu\gamma_5\frac{\tau_i}{2}q(x)|N(p_i)\rangle$$

and take its derivatve

$$\begin{aligned}\partial^\mu \langle N(p_f)|\, A^i_\mu \,|N(p_i)\rangle \;\; &= \;\; im\,\langle N(p_f)|\,\bar{q}\tau^i\gamma_5 q\,|N(p_i)\rangle \\ &= m\,\langle N(p_f)|\,P_i\,|N(p_i)\rangle\end{aligned}$$

But nucleon matrix element of the pseudoscalar density can be related to the pion coupling to the nucleon (Goldberger-Treiman relation, to be discussed later)