QCD lecture 13a

January 4

Functional integral for photons

Here we would naively think that we will have a scalar integral
for each component A_μ However gauge invariance complicates things.
Even more so for QCD.
Let's first write a naive functional integral

$$Z_0[j^\mu] \equiv \int [DA_\mu(x)] \; \exp\left\{ i \int d^4x \left(-\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + j^\mu A_\mu \right) \right\}$$

This is a Gaussian integral, because $F^{\mu\nu}F_{\mu\nu}$ is quadratic in A_μ (exercise)

$$\begin{aligned} -\frac{1}{4}\int d^4x\, F^{\mu\nu}F_{\mu\nu} &= -\frac{1}{4}\int d^4x\, (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= +\frac{1}{2}\int d^4x\, A^\mu (g_{\mu\nu}\Box - \partial_\mu \partial_\nu) A^\nu \\ &= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \tilde{A}^\mu(k)(g_{\mu\nu}k^2 - k_\mu k_\nu)\tilde{A}^\nu(-k) \end{aligned}$$

We need to invert $(g_{\mu\nu}k^2 - k_\mu k_\nu)$ to perform the integral over A_μ

Functional integral for photons

Inverting photonic operator: find α and β

$$\underbrace{(g_{\mu\nu}k^2 - k_\mu k_\nu)\left(\alpha\, g^{\nu\rho} + \beta\, \frac{k^\nu k^\rho}{k^2}\right)}_{\alpha\, k^2 \delta^\rho_\mu - \alpha\, k_\mu k^\rho} = \delta^\rho_\mu$$

This operator is not invertible: some eigenvalues are zero. These flat directions

correspond to the projection of $\widetilde{A}^\mu(k)$ along k^μ

Landau gauge

Decompose $A^\mu = A^\mu_\perp + A^\mu_\parallel$

in the following way:

$$\widetilde{A}^\mu_\perp(k) \equiv \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\widetilde{A}_\nu(k)$$

$$\widetilde{A}^\mu_\parallel(k) \equiv \left(\frac{k^\mu k^\nu}{k^2}\right)\widetilde{A}_\nu(k)\,.$$

The functional measure can be therefore factorized

$$[DA^\mu] = [DA^\mu_\perp]\,[DA^\mu_\parallel]$$

Functional integral for photons

We have $$-\frac{1}{4}\int d^4x\, F^{\mu\nu}F_{\mu\nu} = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, \widetilde{A}^{\mu}(k)\left(g_{\mu\nu}k^2 - k_\mu k_\nu\right)\widetilde{A}^{\nu}(-k)$$

So the $F^{\mu\nu}F_{\mu\nu}$ part is purely transverse, and

$$Z_0[j^\mu] \equiv \int [DA_{\parallel}^{\mu}(x)]\ \exp\left\{i\int d^4x\, j_\mu A_{\parallel}^{\mu}\right\} \times \int [DA_{\perp}^{\mu}(x)]\ \exp\left\{i\int d^4x\, \left(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + j_\mu A_{\perp}^{\mu}\right)\right\}$$

Recall $\widetilde{A}_{\parallel}^{\mu}(k) \equiv \left(\frac{k^\mu k^\nu}{k^2}\right)\widetilde{A}_{\nu}(k)$ but vector current is conserved $k^\mu j_\mu = 0$

and $\int [DA_{\parallel}^{\mu}(x)]$ is an infinite constant that has to be divided out.

When restricted to the transverse directions $g_{\mu\nu}k^2 - k^\mu k^\nu$ is invertible and we get

$$Z_0[j^\mu] = \exp\left\{-\frac{1}{2}\int d^4x d^4y\, j_\mu(x)\, G_F^{0\,\mu\nu}(x,y)\, j_\nu(y)\right\} \qquad G_F^{0\,\mu\nu}(p) \equiv \frac{-i}{p^2 + i0+}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$$

again $i0^+$ prescription selects the ground state for large times.

General covariant gauges

Note that to get $G_F^{0\,\mu\nu}(p) \equiv \frac{-i}{p^2 + i0^+}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$ we demanded $\partial_\mu A^\mu = 0$.

This is called Landau or Lorentz gauge.
In general we may require:

$$\partial_\mu A^\mu(x) = \omega(x)$$

This can be done by introducing a delta function into the functional integral

$$Z_0[j^\mu] \equiv \int [D\omega(x)]\ \exp\left\{-i\frac{\xi}{2}\int d^4x\ \omega^2(x)\right\}$$
$$\times \int [DA_\mu(x)]\ \delta[\partial_\mu A^\mu - \omega]\ \exp\left\{i\int d^4x\ \left(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + j^\mu A_\mu\right)\right\}$$

where ξ is an arbitrary constant. Note that for fixed ω we break Lorentz invariance. To mitigate this problem we integrate over all ω's with the Gaussian weight. We can do this Gaussian integral and integrating by parts (exercise) we arive at

$$Z_0[j^\mu] = \int [DA_\mu(x)]\ \exp\left\{i\int d^4x\ \left(\tfrac{1}{2}A^\mu(g_{\mu\nu}\Box - (1-\xi)\partial_\mu\partial_\nu)A^\nu + j^\mu A_\mu\right)\right\}$$

We need to find inverse of $i(g_{\mu\nu}p^2 - (1-\xi)p_\mu p_\nu)$

General covariant gauges

To invert $i(g_{\mu\nu}p^2 - (1-\xi)p_\mu p_\nu)$

we look for the inverse operator in a form: $\alpha\, g^{\nu\rho} + \beta\, \frac{p^\nu p^\rho}{p^2}$

The result reads (exercise)

$$G_F^{0\,\mu\nu}(p) = \frac{-i\,g^{\mu\nu}}{p^2 + i0^+} + \frac{i}{p^2 + i0^+}\left(1 - \frac{1}{\xi}\right)\frac{p^\mu p^\nu}{p^2}$$

Landau gauge: $\xi \to \infty$

Feynman gauge: $\xi = 1$

As we will see in QCD the gauge condition will be more like $F(\partial_\mu A^\mu) - \omega \;= 0$
and then we will need a Jacobian (to be discussed later)

$$\int [D\omega(x)]\ \exp\left\{-i\frac{\xi}{2}\int d^4x\ \omega^2(x)\right\}\int [DA_\mu(x)]\ \underbrace{F'(\partial_\mu A^\mu)}_{\text{Jacobian}}\ \delta[F(\partial_\mu A^\mu) - \omega]$$