QCD lecture 12

December 14

Anomaly – summary from lect. 11

Chiral transformation $\mathcal{U}(x) = e^{i\alpha(x)\gamma^5 t}$ changes fermionic integration measure:

$$[D\psi D\overline{\psi}] \to \frac{1}{(\det \mathcal{U})^2}\,[D\psi D\overline{\psi}]$$

This change can be effectively added to action $\mathcal{L}(x) \to \mathcal{L}(x) + \alpha(x)\mathcal{A}(x)$

$$(\det \mathcal{U})^{-2} = e^{-2\,\mathrm{tr}\,\ln \mathcal{U}} \underset{\alpha \ll 1}{\approx} \exp\left[i\int d^4x\ \alpha(x)\mathcal{A}(x)\right]$$

where $\mathcal{A}(x) \equiv -2\,\mathrm{tr}\,(\gamma^5 t)\,\delta(x-x)$

This expression is mtemthically not well defined. Fujikawa proposed the following gauge invariant regularization:

$$\mathcal{A}(x) = -2 \lim_{y\to x, M\to +\infty} \mathrm{tr}\left\{\gamma^5\, t\, \mathcal{F}\left(-\frac{\not{D}_x^2}{M^2}\right)\right\}\delta(x-y)$$

where $\not{D}_x \equiv \gamma^\mu\left(\partial_\mu - i\,g\,t^a A^a_\mu(x)\right)$

Anomaly – summary from lect. 11

Regularized expression can be computed:

$$\mathcal{A}(x) = -\frac{g^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma} F_a^{\mu\nu}(x) F_b^{\rho\sigma}(x) \, \mathrm{tr}\,(t^a t^b t)$$

This is the same expression we got in perturbation theory. Note that Fujicawa method is explicitely gauge invariant, while in perturbative calculation we had to impose vector current conservation.

In Euclidean metric Dirac operator is hermitean and we can relate anomaly to the number of zero modes (Atiyah-Singer index theorem):

$$\frac{g^2}{32\pi^2}\int d^4x_E \; \epsilon_{ijkl} F_{ij}^a(x) F_{kl}^b(x) \, \mathrm{tr}(t^a t^b) = n_R - n_L$$

where

$$\not{D}_x \phi_R(x) = 0\,, \quad \gamma^5 \phi_R(x) = +\phi_R(x)$$
$$\not{D}_x \phi_L(x) = 0\,, \quad \gamma^5 \phi_L(x) = -\phi_L(x)$$

θ term and strong CP problem

Recall QCD Lagrangian: $$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right] + \sum_{f=1}^{6}\left[\overline{q}_f\, i\gamma^\mu D_\mu q_f - m_f \overline{q}_f\, q_f\right]$$

In principle we could add a new term that has the same dimension as $\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}$

$$\mathcal{L}_\theta \equiv \frac{g^2\theta}{32\pi^2}\,\epsilon_{\mu\nu\rho\sigma}\;\mathrm{tr}\left(F^{\mu\nu}\,F^{\rho\sigma}\right) \;=\frac{g^2\theta}{32\pi^2}\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$$

called the θ-term. This is precisely the anomaly multiplied by a dimesionless coupling constant θ. This term, however, can be expressed as a total derivative and therefore does not contribute to the equations of motion:

$$\partial_\mu K^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\,F^a_{\mu\nu}\,F^a_{\rho\sigma}$$

where (exercise)

$$K^\mu \equiv \epsilon^{\mu\nu\rho\sigma}\left[A^a_\nu F^a_{\rho\sigma} - \frac{g}{3}\,f^{abc}\,A^a_\nu A^b_\rho A^c_\sigma\right]$$

or

$$K^\mu \equiv 2\epsilon^{\mu\nu\rho\sigma}\;\mathrm{tr}\left[A_\nu F_{\rho\sigma} + \frac{2ig}{3}A_\nu A_\rho A_\sigma\right]$$

and $\mathcal{L}_\theta = \frac{g^2\theta}{32\pi^2}\,\partial_\mu K^\mu$ This term is Lorentz and (small) gauge invariant but violates CP.

θ term and strong CP problem

θ term is related to the neutron electric dipole moment: $|\theta| \lesssim 10^{-10}$

Why is it so small? One would naturally expect θ ~ 1. This is called strong CP problem.

Relation to the quark masses

$$\psi_f \longrightarrow e^{i\gamma_5 \alpha_f} \psi_f$$

This transformation is anomalous

$$[D\psi D\overline{\psi}] \longrightarrow \exp\left(-\frac{i}{32\pi^2}\int d^4x\, \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \sum_f \alpha_f\right) [D\psi D\overline{\psi}]$$

The same effect would be caused by a change of coupling θ
(if we add theta term to the lagrangian)

$$\theta \to \theta - 2\sum_f \alpha_f$$

θ term and strong CP problem

Let's allow for the complex fermion masses: this would violate P and CP

$$\sum_f M_f\, \overline{\psi}_f \frac{1+\gamma_5}{2}\psi_f + \sum_f M_f^*\, \overline{\psi}_f \frac{1-\gamma_5}{2}\psi_f$$

Transformation $\psi_f \longrightarrow e^{i\gamma_5 \alpha_f}\psi_f$ results in (exercise)

$$\sum_f e^{2i\alpha_f} M_f\, \overline{\psi}_f \frac{1+\gamma_5}{2}\psi_f + \sum_f e^{-2i\alpha_f} M_f^*\, \overline{\psi}_f \frac{1-\gamma_5}{2}\psi_f$$

which is equivalent to $M_f \to e^{2i\alpha_f} M_f$

Since any change of θ can be undone by a chiral transformation of quarks physical quantities cannot depend separately on θ and M_f but on the combination:

$$e^{i\theta}\prod_f M_f$$

which is invariant. So θ term would have no effect if at least one quark mass was zero.

Possible solution to the CP problem – axion: θ is a field (not discussed here)

Topology of gauge fields

Since $\mathcal{L}_\theta$ is a full derivative, we can apply Stokes' theorem to calculate the action

$$\int d^4x_E\, \mathcal{L}_\theta = \frac{g^2\theta}{32\pi^2}\int d^4x_E\, \partial_\mu K^\mu = \frac{g^2\theta}{32\pi^2}\lim_{R\to\infty}\int_{S_{3,R}} dS_\mu\, K^\mu$$

3-dim sphere of radius R

Recall non-Abelian gauge transformation (now we use Ω rather than U):

$$A_\mu(x) \quad \to \quad A_\mu^\Omega(x) \equiv \Omega^{-1}(x)\,A_\mu(x)\,\Omega(x) + \frac{i}{g}\,\Omega^{-1}(x)\big(\partial_\mu\Omega(x)\big)$$

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pure gauge

If all color sources are placed in a finite region of space time, we can assume that the gauge fields on the 3-sphere are pure gauge plus a small correction:

$$A_\mu(x) = a_\mu(x) + \frac{i}{g}\,\Omega^\dagger(\widehat{x})\,\partial_\mu\,\Omega(\widehat{x})$$

and matrix $\Omega(\widehat{x})$ depends only on the direction of x^μ Since $\dfrac{\partial}{\partial x^\mu} = \dfrac{1}{|x|}\dfrac{\partial}{\partial \hat{x}^\mu}$

$$A_\mu \to \frac{1}{|x|} \text{ for } |x| \to \infty$$

Topology of gauge fields

If $\quad A_\mu \to \frac{1}{|x|}$ for $|x| \to \infty$

then

$$K^\mu \equiv 2\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left[A_\nu F_{\rho\sigma} + \frac{2ig}{3}A_\nu A_\rho A_\sigma\right] \underset{|x|\to+\infty}{\longrightarrow} \frac{4ig}{3}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(A_\nu A_\rho A_\sigma\right) \sim |x|^{-3}$$

One can show that $F_{\rho\sigma}(x)$ for pure gauge is zero (exercise)

Therefore

$$\int d^4x_E\,\mathcal{L}_\theta = \frac{\theta}{24\pi^2}\lim_{R\to\infty}\int_{\mathcal{S}_{3,R}} dS\,\widehat{x}_\mu\,\epsilon^{\mu\nu\rho\sigma} \times\mathrm{tr}\left(\Omega^\dagger(\partial_\nu\Omega)\Omega^\dagger(\partial_\rho\Omega)\Omega^\dagger(\partial_\sigma\Omega)\right)$$

Since $dS \sim R^3$ the integral is finite and we can drop $\lim$ Therefore the integral depends only on $\Omega(\widehat{x})$ - unitary matrix that maps a 3-dim sphere in Euclidean space-time onto the gauge group

$$\Omega : \quad \mathcal{S}_3 \quad \longmapsto \quad \mathcal{G}$$

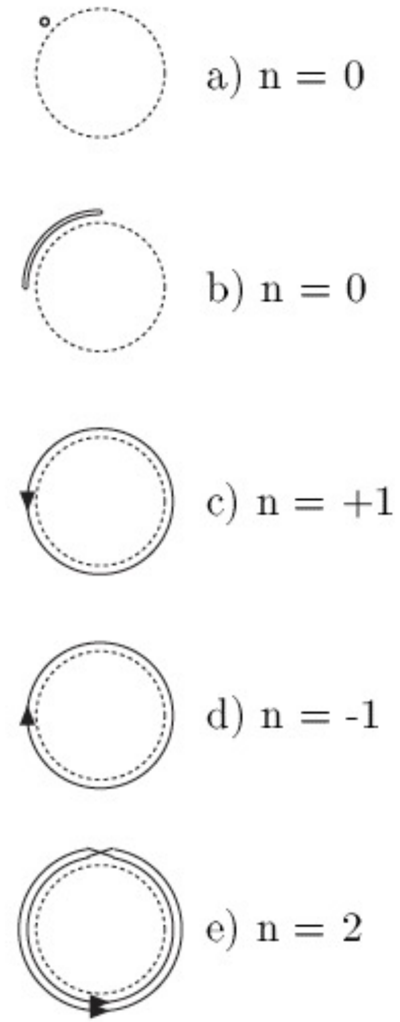
Topology of mappings

Consider baby-model: mapping of
1 dim sphere (circle) onto U(1) group,
which is also a circle.

One can characterize these mappings by
a winding number.

Mappings from one class cannot be deformed
into a mapping of another class. They are called
homotopy classes:

$$\pi_1(\mathrm{U}(1)) = \mathbb{Z}$$



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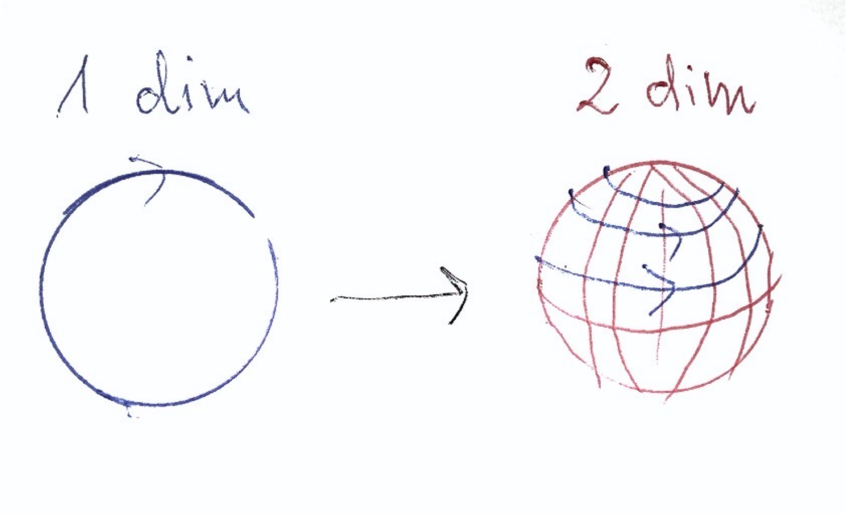
Topology of mappings

Some mappings can be always shrunk to a point.



Homotopy classes

We have a mapping $S^3_{space} \rightarrow SU\left(3\right)_{color}$

To discuss topology it is convenent to restrict discussion to an SU(2) subgroup of SU(N)

For $U \in SU\left(2\right)$ we have the following parametrization $U = u_0 + i u_a \tau_a$
with u_α real satisfying $u_0^2 + u_a u_a = 1$ But this is equation of a 3-sphere!

So in practice we have the following mapping

$$S^3_{space} \rightarrow S^3_{group}$$

It is known (generalization of our 1 dim example)

$$\pi_d\left(S^d\right) = Z$$

Homotopy classes

Mapping of a 3 dimensional sphere is characterized by homotopy class $\pi_3(\mathcal{G})$

So for SU(*N*) where *N* > 1

$$\pi_3(\mathrm{SU}(N)) = \mathbb{Z}$$

We see now that anomaly, that is an integer

$$\frac{g^2}{32\pi^2}\int d^4x_E\ \epsilon_{ijkl}\, F^a_{ij}(x)\, F^b_{kl}(x)\,\mathrm{tr}(t^a t^b) = n_R - n_L$$

is related to the topology of gauge fields.

Now we can understand notation $\langle \partial_\mu J^\mu_5(x)\rangle_A = -\frac{g^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\, F^{\mu\nu}_a(x)\, F^{\rho\sigma}_b(x)\,\mathrm{tr}\,(t^a t^b t)$;

Instantons - preliminaries

Consider QCD in temporal gauge $A_0^a = 0$ There are still residual time-independent gauge transformations possible that preserve this condition denoted by U
We shall assume that they approach a constant at spacial infinty, chosen to be unity (vacuum):

$$U(\vec{x}) \to 1 \qquad \text{for} \qquad |\vec{x}| \to \infty$$

This means that all points at spacial infity correspond to the same value of U, so we can identify them (squeeze to a point), which means that $R^3{}_{space}$ compactified to a sphere, so that we have a mapping

$$S^3_{space} \to SU(3)_{color}$$

These mappings fall into a distinct tolological classes characterized by an integer n

$$U^{(n)}(x)$$

For pure gauge field

$$A_\mu^{(n)} = -\frac{i}{g} U^{(n)} \partial_\mu U^{(n)\dagger}$$

(field tensor is zero! – exercise) in a given class n we cannot penetrate to another class m within a pure gauge configuration

Instantons - preliminaries

In order to continously deform $A_\mu^{(n)} \to A_\mu^{(m)}$ we have to consider field configurations with nonminimal action $S_E > 0$

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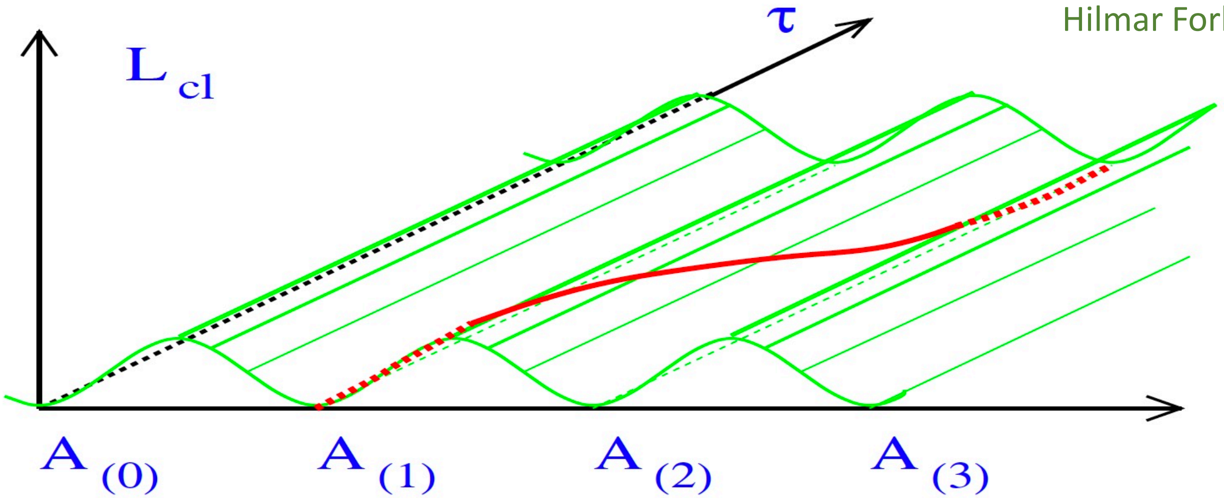
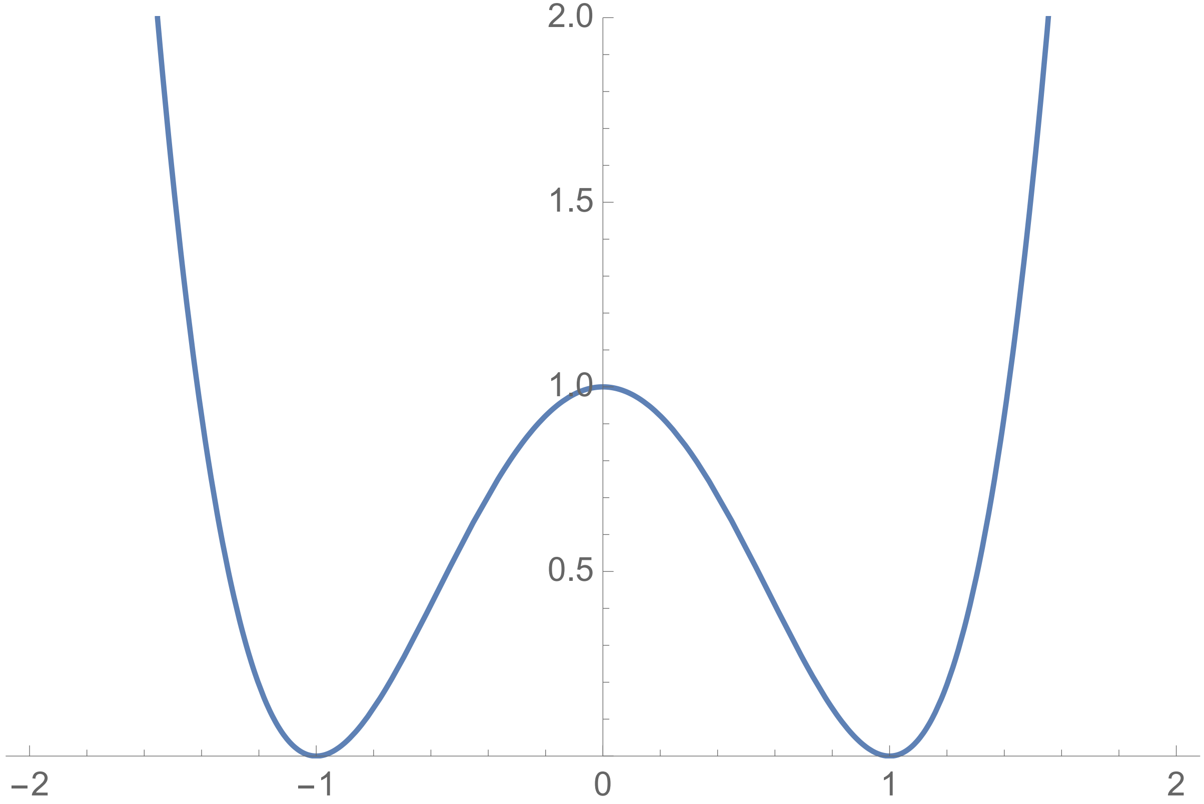


$$n = \frac{1}{24\pi^2}\int d^3\mathbf{x}\, \epsilon^{ijk}\left[(U^\dagger \partial_i U)(U^\dagger \partial_j U)(U^\dagger \partial_k U)\right]$$

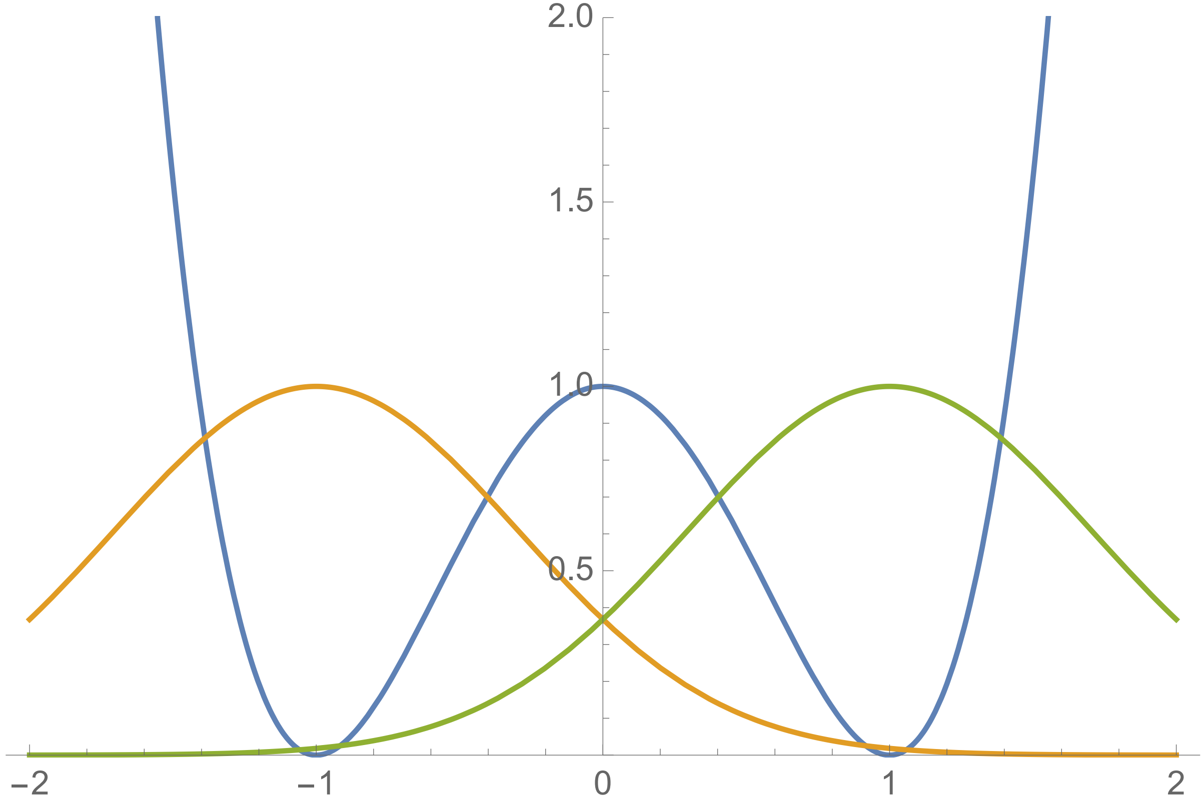
Example (hedgehog) $\quad U = \exp[i(r\cdot\tau)/r\, P(r)] \quad P(0) = n\pi, \;\; P(\infty) = 0$

Exercise: calculate n

Double well potential

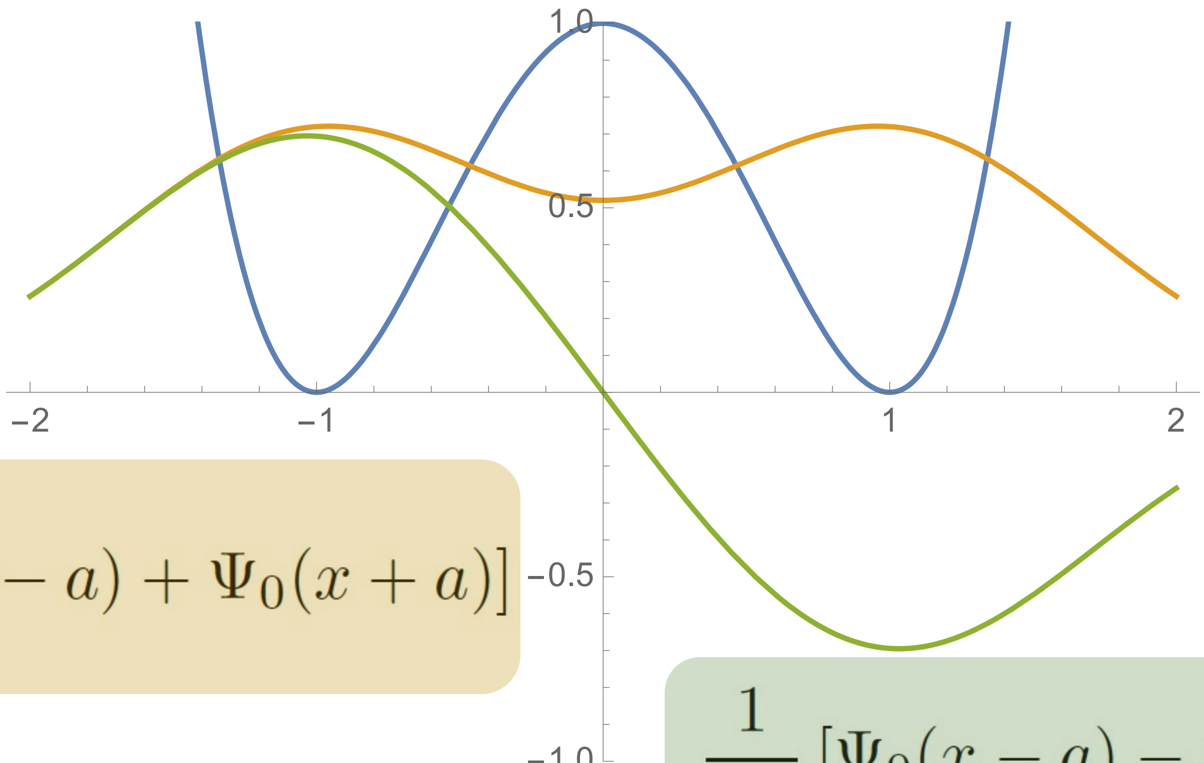


Double well potential



Two (almost) degenarate states: one concentrated around -1, the other one around +1. However since there is tunneling we expect two nearly dgenarte lowest energy states.

Double well potential



$$\frac{1}{\sqrt{2}}\left[\Psi_0(x-a)+\Psi_0(x+a)\right]$$

$$\frac{1}{\sqrt{2}}\left[\Psi_0(x-a)-\Psi_0(x+a)\right]$$

Goal: calculate the energy splitting using path integral formalism.

Calculate $K(a,\text{-}a,T)$ and use energy representation $K(a,-a,T)=\sum_n e^{-i\frac{E_n T}{\hbar}}\phi_n(a)\phi_n^*(-a)$

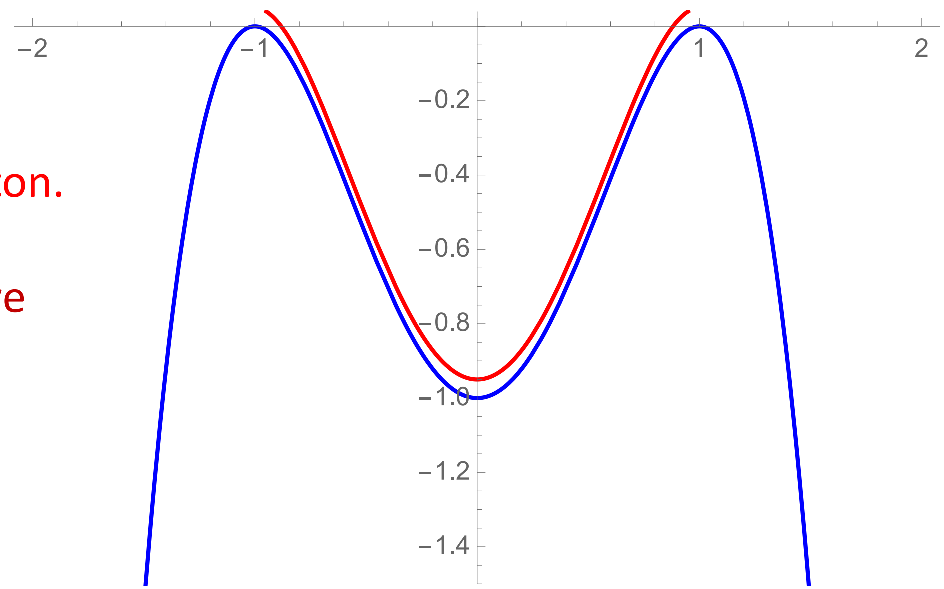
Euclidean path integral

There is no classical trajectory: $-a \to a$ Go to Euclidean time $t = -i\tau$
where

$$K_E(x_b, \tfrac{1}{2}T; x_a, -\tfrac{1}{2}T) = <x_a|e^{-\frac{1}{\hbar}HT}|x_a> = \int [\mathcal{D}_E x(\tau)] e^{-\frac{1}{\hbar}S_E[x(\tau)]}$$

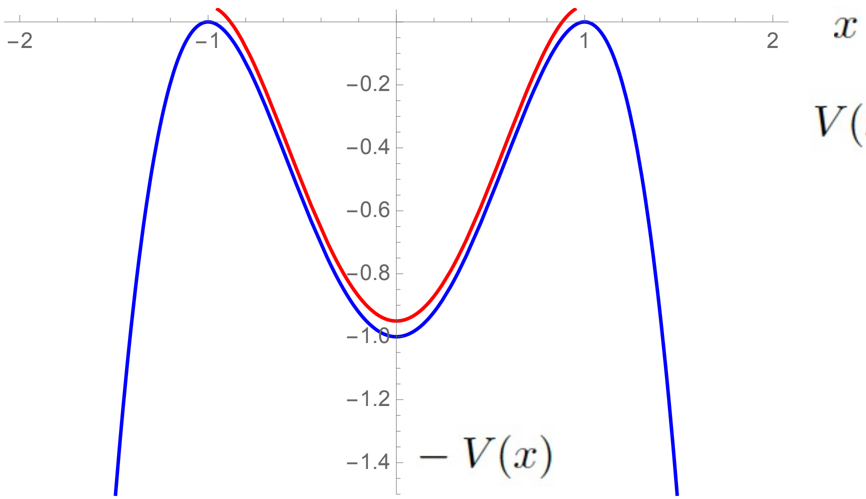
$$S_E[x(\tau)] = \int_{-T/2}^{T/2} d\tau \left[\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]$$

Potential is inverted and there is a classical trajectory called instanton.



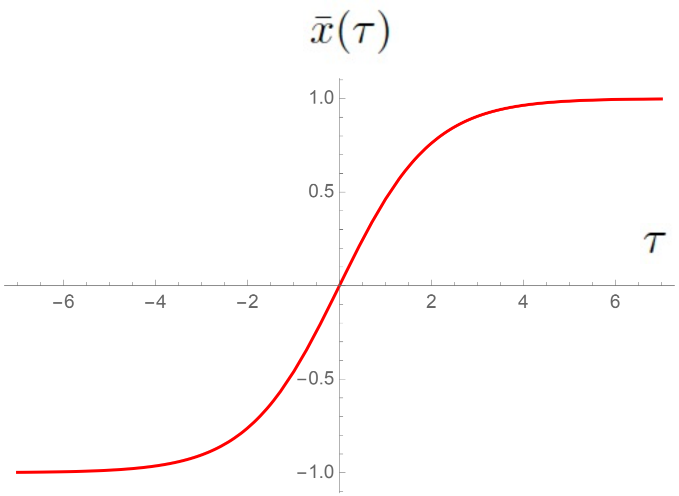
To calculate the energy splitting we have to sum over an infinite number of instantons

Explicit model



$$V(x) = \frac{1}{8a^2}(a^2 - x^2)^2$$

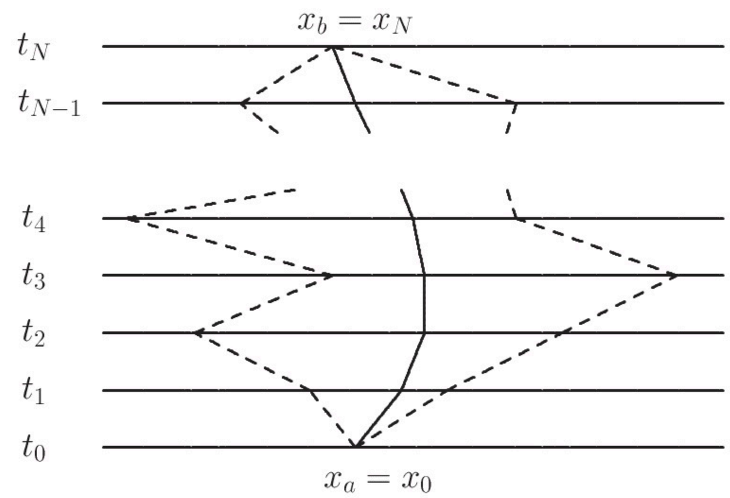
Instanton is an Euclidean trajectory of zero energy.



$$|\bar{x} - a| \sim e^{-\sqrt{\frac{V''(a)}{m}}\tau} = e^{-\omega\tau}$$

$$\bar{x}(\tau) = a \tanh\frac{\tau - \tau_1}{2}$$

Path integral in QM – reminder



$$K(x_b, x_a, t_b - t_a) = F(t_b - t_a)\, e^{\frac{i}{\hbar} S[\bar{x}(t)]}$$

$$\delta^2 S \;=\; -\int_0^T y\left[\frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{x}^2}\dot{y}\right) + \frac{d}{dt}\left(\frac{\partial^2 L}{\partial x \partial \dot{x}}\right) y - \frac{\partial^2 L}{\partial x^2} y\right] dt \;=\; \int_0^T y\, D(t) y\, dt\,.$$

D is a Sturm-Liouville operator $\quad D(t) y_n(t) \;=\; \lambda_n y_n(t)\,, \quad n = 1, 2, 3, \ldots\,, \quad \lambda_1 < \lambda_2 < \ldots$

Use y_n basis to expand $\quad y(t) = \sum_{n=1}^{\infty} a_n\, y_n(t)$ then $\quad \delta^2 S[y] = \sum_{n=1}^{\infty} \lambda_n a_n^2$

and $\quad [\mathcal{D} y(t)] \sim \prod_{n=1}^{\infty} da_n$

$$F(T) \sim \prod_{n=1}^{\infty} da_n \, \exp\left(\frac{i}{2\hbar}\lambda_n a_n^2\right) \sim \sqrt{\frac{1}{\prod_n \lambda_n}} = \sqrt{\frac{1}{\det D(t)}}$$

Instanton – classical action

Recall that instanton has $E = 0$

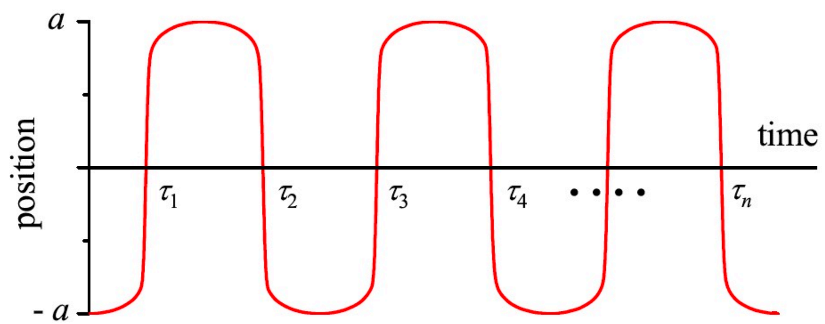
$$\tfrac{1}{2}m\dot{\bar{x}}^2 - V(x) = 0\,, \qquad \dot{\bar{x}} = \left[\frac{2}{m}V(\bar{x})\right]^{\frac{1}{2}}, \qquad \frac{d\tau}{d\bar{x}} = \frac{1}{\sqrt{\frac{2}{m}V(\bar{x})}}$$

Hence

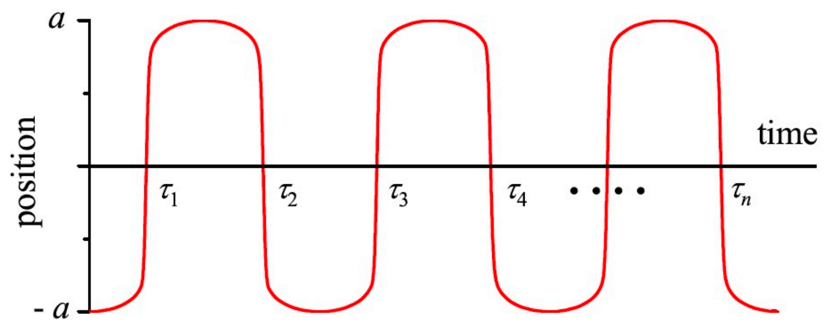
$$S_E^0 = \int_{-T/2}^{+T/2} d\tau \left[\tfrac{1}{2}m\frac{2}{m}V(\bar{x}) + V(\bar{x})\right] = \int_{-a}^{+a} d\bar{x}\sqrt{2mV(\bar{x})} = \int_{-a}^{+a} d\bar{x}\, p(\bar{x})$$

barrier transmission coefficient

Consider now amplitude $<\!-a|e^{-HT/\hbar}|-a\!>$ that has infnitely many jumps: instantons and anti-instantons separated in time (dilute approximation)



Multi-instanton transition amplitude



$$x(\tau) = \bar{x}_{\tau_1 \dots \tau_n}(\tau) + y(\tau) \approx \bar{x}_{\tau_1}(\tau) + \bar{x}_{\tau_2}(\tau) + \dots + \bar{x}_{\tau_n}(\tau) + y(\tau)$$

Here $\bar{x}_{\tau_1 \dots \tau_n}(\tau)$ is the exact classical trajectory that can be approximated by a sum over one-(anti) instanton trajectories $\bar{x}_{\tau_n}$ where $\tau_1, \dots, \tau_n$ mark times of individual jumps.

$$< -a | e^{-\frac{1}{\hbar}HT} | -a > = \sum_{\text{even}\, n} \int_{-T/2}^{T/2} d\tau_1 \dots \int_{\tau_{n-2}}^{T/2} d\tau_{n-1} \int_{\tau_{n-2}}^{T/2} d\tau_n \, e^{-\frac{1}{\hbar} S_E[\bar{x}_{\tau_1 \dots \tau_n}(\tau)]}$$

$$\times \int_{y(-\frac{T}{2}) = y(+\frac{T}{2}) = 0} [\mathcal{D} y_E(\tau)] \, e^{-\frac{1}{2\hbar} \int_{-T/2}^{+T/2} d\tau y(-m \frac{d^2}{d\tau^2} + V''(\bar{x}))y}$$

Multi-instanton transition amplitude

In dilute approximation $S_E[\bar{x}_{\tau_1\ldots\tau_n}(\tau)] \approx nS_E^0$

The quantal part can be written as a kind of propagator

$$K_E(0,\tfrac{1}{2}T;0,-\tfrac{1}{2}T) = \int_{y(-\frac{T}{2})=y(+\frac{T}{2})=0} [\mathcal{D}y_E(\tau)]\, e^{-\frac{1}{2\hbar}\int\limits_{-T/2}^{+T/2} d\tau\, y(-m\frac{d^2}{d\tau^2}+V''(\bar{x}))y}$$