QCD lecture 10

November 30

Functional integral for scalar field

One can easily translate the QM functional formalism to QFT with the help of the following correspondence

$$\begin{aligned} q(t) &\longleftrightarrow \phi(x) \\ p(t) &\longleftrightarrow \Pi(x) \\ j(t) &\longleftrightarrow j(x) \end{aligned}$$

and the analogue of the generating functional reads

$$Z[j] = \int [D\Pi(x) D\phi(x)] \times \exp\left\{ i \int d^4x \, \left(\Pi(x)\dot{\phi}(x) - (1 - i0^+)\mathcal{H}(\Pi, \phi) + j(x)\phi(x) \right) \right\}$$

The hamiltonian reads

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)\cdot(\nabla\phi) + \frac{1}{2}m^2\phi^2 + V(\phi)$$

and can be obtained from the Lagrangian

$$\mathcal{L} = \int d^3\mathbf{x} \, \left\{ \tfrac{1}{2}(\partial_\mu \phi(x))(\partial^\mu \phi(x)) - \tfrac{1}{2}m^2\phi^2(x) \right\}$$

Functional integral for scalar field

Since the hamiltonian is quadratic in Π we can perform Gaussian integral

$$Z[j] = \int [D\phi(x)] \; \exp\left\{ i \int d^4x \, \left(\mathcal{L}(\phi) + j(x)\phi(x)\right)\right\}$$

where

$$\mathcal{L}(\phi) \equiv \frac{1}{2}(1+i0^+)\dot{\phi}^2 - \frac{1}{2}(1-i0^+)\big((\nabla\phi)\cdot(\nabla\phi) + m^2\phi^2\big) - (1-i0^+)V(\phi)$$

Note that $1 - i0^+$ in front of V plays no role if interaction vanishes for large times.
Then

$$Z[j] = \exp\left\{ -i \int d^4x \, V\Big(\frac{\delta}{i\delta j(x)}\Big)\right\} Z_0[j]$$

where

$$Z_0[j] \equiv \int [D\phi(x)] \; \exp\left\{ i \int d^4x \, \left(\mathcal{L}_0(\phi) + j(x)\phi(x)\right)\right\}$$

and

$$\mathcal{L}_0(\phi) = \frac{1}{2}(1+i0^+)\dot{\phi}^2 - \frac{1}{2}(1-i0^+)\big((\nabla\phi)\cdot(\nabla\phi) + m^2\phi^2\big)$$

Fermions and Grassmann variables

Hermann Günther Grassmann (1809 Szczecin – 1877 Szczecin)

Fermion fields anticommute. How to take this into account in functional integral?
Introduce Grassmann variables:

$$\psi_i \ (i = 1 \cdots N)$$

$$\{\psi_i, \psi_j\} = 0$$

Linear space spanned by ψ_i's is called Grassmann algebra

Consider first $N = 1$ $\quad \psi^2 = 0$

any function has a form $f(\psi) = a + \psi b$ where a is a number and $\{b, b\} = \{b, \psi\} = 0$

so $f(\psi) = a + \psi b = a - b\psi$

We have to define left and right derivatives $\overrightarrow{\partial}_\psi \, f(\psi) = b \quad , \qquad f(\psi) \, \overleftarrow{\partial}_\psi = -b$

Berezin integral: $\int d\psi \; \alpha f(\psi) = \alpha \int d\psi \; f(\psi)$ and $\int d\psi \; \partial_\psi f(\psi) = 0$

The only solution consistent with these requirements $\int d\psi \; f(\psi) = b$

Fermions
and Grassmann variables

Hermann Günther Grassmann (1809 Szczecin – 1877 Szczecin)

Fermion fields anticommute. How to take this into account in functional integral?
Introduce Grassmann variables:

$$\psi_i \; (i = 1 \cdots N)$$

$$\{\psi_i, \psi_j\} = 0$$

Linear space spanned by ψ_i's is called Grassmann algebra

Consider first $N = 1$ $\psi^2 = 0$

any function has a form $f(\psi) = a + \psi b$ where a is a number and $\{b, b\} = \{b, \psi\} = 0$

so $f(\psi) = a + \psi b = a - b\psi$

We have to define left and right derivatives $\overrightarrow{\partial}_\psi \; f(\psi) = b$, $f(\psi) \; \overleftarrow{\partial}_\psi = -b$

Berezin integral: $\int d\psi \; \alpha f(\psi) = \alpha \int d\psi \; f(\psi)$ and $\int d\psi \; \partial_\psi f(\psi) = 0$

The only solution consistent with these requirements $\int d\psi \; f(\psi) = b$

$$\int d\psi \; 1 = 0$$

$$\int d\psi \; \psi = 1$$

Functions of Grassmann variables

Consider now N Grassmann variables $\psi \equiv (\psi_1, \cdots, \psi_N) \quad \{\psi_i, \psi_j\} = 0$

The most general function: $f(\psi) = \sum_{p=0}^{N} \frac{1}{p!} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_p} C_{i_1 i_2 \cdots i_p}$

Only linear terms in each variable are possible. Note that it must be $C_{i_1 \cdots i_N} \equiv \gamma\, \epsilon_{i_1 \cdots i_N}$

alternatively $\frac{1}{N!} \psi_{i_1} \cdots \psi_{i_N}\, \gamma\, \epsilon_{i_1 \cdots i_N} = \psi_1 \cdots \psi_N\, \gamma$

For consitency with previous definition $\int d^N\psi\, f(\psi) = \gamma$

Terms where at least one variable is missing do not contribute to the integral because $\int d\psi\, 1 = 0$

Integration measure $d^N\psi \equiv d\psi_N\, d\psi_{N-1} \cdots d\psi_1$ assures that

$$\int d^N\psi\, \psi_1 \cdots \psi_N = \int d\psi_N \cdots \Big(\underbrace{\int d\psi_2 \Big(\underbrace{\int d\psi_1\, \psi_1}_{1} \Big) \psi_2}_{1} \Big) \cdots \psi_N = 1$$

Change of variables

Consider $\psi_i \equiv J_{ij}\,\theta_j$ where $\theta_1 \cdots \theta_N$ are also Grassmann variables

Last term of the function $f(\psi)$

$$\begin{aligned}\psi_{i_1} \cdots \psi_{i_N}\, \epsilon_{i_1 \cdots i_N}\, \gamma &= (J_{i_1 j_1}\theta_{j_1}) \cdots (J_{i_N j_N}\theta_{j_N})\, \epsilon_{i_1 \cdots i_N}\, \gamma \\ &= \det(J)\, \theta_{j_1} \cdots \theta_{j_N}\, \epsilon_{j_1 \cdots j_N}\, \gamma\,.\end{aligned}$$

From this we conclude

$$\underbrace{\int d^N\psi\, f(\psi)}_{\gamma} = [\det(J)]^{-1} \underbrace{\int d^N\theta\, f(\psi(\theta))}_{\det(J)\,\gamma}$$

(same as for scalar integral)

Gaussian integral

Consider $\psi \equiv (\psi_1, \cdots, \psi_N) \quad \{\psi_i, \psi_j\} = 0$

$$I(\mathbf{M}) \equiv \int d^N\psi \ \exp\left(\tfrac{1}{2}\psi_i M_{ij}\psi_j\right)$$

where M is $N \times N$ antisymmetric numeric matrix (real or complex). Such integral is non-zero only if N is even. For $N = 2$

$$\mathbf{M} = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}$$

and $\exp\left(\tfrac{1}{2}\psi_i M_{ij}\psi_j\right) = 1 + \mu\,\psi_1\psi_2$

Hence: $I(\mathbf{M}) = \mu = \left[\det(\mathbf{M})\right]^{1/2}$

For general even N we can always "diagonalize" $\mathbf{M}$ by special orthogonal matrix

$$\mathbf{M} = \mathbf{Q}\underbrace{\begin{pmatrix} 0 & \mu_1 & & & \\ -\mu_1 & 0 & & & \\ & & 0 & \mu_2 & \\ & & -\mu_2 & 0 & \\ & & & & \ddots \end{pmatrix}}_{\mathbf{D}}\mathbf{Q}^T$$

Define $\mathbf{Q}^T\psi \equiv \theta,$

Gaussian integral

After change of variables we get

$$I(\mathbf{M}) = \left[\det(\mathbf{Q})\right]^{-1} \underbrace{\int d^N\theta \ \exp\left(\tfrac{1}{2}\,\theta^{\top}\mathbf{D}\theta\right)}_{\mu_1\mu_2\cdots=[\det(\mathbf{D})]^{1/2}}$$

But $\det(\mathbf{Q}) = +1$ and we have

$$I(\mathbf{M}) = \left[\det(\mathbf{D})\right]^{1/2} = \left[\det(\mathbf{M})\right]^{1/2}$$

This is inverse with respect to the Gaussian integral for ordinary variables

We will also need integrals with Grassmann sources η_i

$$I(\mathbf{M}, \eta) \equiv \int d^N\psi \ \exp\left(\tfrac{1}{2}\,\psi_i M_{ij}\psi_j + \eta_i\psi_i\right)$$

Changing variables

$$\psi_i' \equiv \psi_i - M_{ij}^{-1}\eta_j$$

we obtain

$$I(\mathbf{M}, \eta) = \left[\det(\mathbf{M})\right]^{1/2} \ \exp\left(-\tfrac{1}{2}\eta^{\top}\mathbf{M}^{-1}\eta\right)$$

Gaussian integral for 2N variables

Consider $J(\mathbf{M}) \equiv \int d^N \xi d^N \psi \ \exp\left(\psi_i M_{ij} \xi_j\right)$ where ψ and ξ are independent

Then (exercise) $J(\mathbf{M}) = \det\left(\mathbf{M}\right)$

Complex Grassmann variables

Define $\chi \equiv \frac{\psi + i\xi}{\sqrt{2}}$, $\overline{\chi} \equiv \frac{\psi - i\xi}{\sqrt{2}}$ and inverse $\psi = \frac{\overline{\chi} + \chi}{\sqrt{2}}$, $\xi = \frac{i\left(\overline{\chi} - \chi\right)}{\sqrt{2}}$

Integrations

$$d\xi d\psi = i\, d\chi d\overline{\chi}\,,$$

$$\psi\xi = -i\,\overline{\chi}\chi\,,$$

$$\int d\chi d\overline{\chi}\ \overline{\chi}\chi = \int d\xi d\psi\ \psi\xi = 1$$

which leads to $\int d\chi d\overline{\chi}\ \exp\left(\mu\,\overline{\chi}\chi\right) = \mu$

or generally

$$\int d\chi_N\, d\overline{\chi}_N \cdots d\chi_1\, d\overline{\chi}_1\ \exp(\overline{\chi}^T \mathbf{M} \chi) = \det\left(\mathbf{M}\right)$$

with sources

$$\int d\chi_N\, d\overline{\chi}_N \cdots d\chi_1\, d\overline{\chi}_1\ \exp\left(\overline{\chi}^T \mathbf{M} \chi + \overline{\eta}^T \chi + \overline{\chi}^T \eta\right) = \det\left(\mathbf{M}\right)\ \exp\left(-\overline{\eta}^T \mathbf{M}^{-1} \eta\right)$$

Functional integral for fermions

The functional integral reads

$$Z_0[\bar{\eta},\eta] = \int [D\psi(x)D\overline{\psi}(x)] \exp\left\{ i\int d^4x \Big(\overline{\psi}(x)(i\partial\!\!\!/ - m)\psi(x) + \bar{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x) \Big) \right\}$$

Fermionic symmetries

Consider a set of fermion fields $\psi(x)$ (with components $\psi_n(x)$) interacting with a gauge potential $A_\mu^a(x)$

At this moment we do not specify the meaning of index *n*

- it can be color index
- it can be flavor (up, down, strange ...)
- it can be spinor index
- or some combination of the above

What is important is the unitary transformation of these fields

$$\psi(x) \to U(x)\psi(x) \qquad \psi^\dagger(x) \to \psi^\dagger(x)U^\dagger(x)$$

Dirac conjugate: $\overline{\psi}(x) \equiv \psi^\dagger(x)\gamma^0 \to \psi^\dagger(x)U^\dagger(x)\gamma^0 = \overline{\psi}(x)\underline{\gamma^0 U^\dagger(x)\gamma^0}$

These matrices have both fermionic and space-time indices: $\overline{\mathcal{U}}$

$$\mathcal{U}_{xm,yn} \equiv U_{mn}(x)\,\delta(x-y)\,,$$
$$\overline{\mathcal{U}}_{xm,yn} \equiv (\gamma^0 U^\dagger(x)\gamma^0)_{mn}\,\delta(x-y)$$

Fermionic symmetries

EXAMPLE:

$$U(x) = e^{i\alpha(x)t}$$

where $\alpha(x) \in \mathbb{R}$ and t is hermitean matrix (generator) that carries no spinor indices.

Then multiplication is understood as (remember that $(\gamma^0)^2 = 1$):

$$\begin{aligned}
(\overline{\mathcal{U}}\mathcal{U})_{xm,yn} &= \int d^4z \sum_p \overline{\mathcal{U}}_{xm,zp}\, \mathcal{U}_{zp,yn} \\
&= \int d^4z\, \delta(x-z)\delta(z-y) \sum_p \left(e^{-i\alpha(z)t}\right)_{mp} \left(e^{i\alpha(z)t}\right)_{pn} \\
&= \delta_{mn}\delta(x-y)\,.
\end{aligned}$$

This means that $\overline{\mathcal{U}}\mathcal{U} = 1$, which implies that $\det \mathcal{U} \det \overline{\mathcal{U}} = 1$

Since under such transformation the Grassmann integration measure changes as

$$[D\psi D\overline{\psi}] \to \frac{1}{\det(\mathcal{U})\, \det(\overline{\mathcal{U}})}\, [D\psi D\overline{\psi}]$$

the measure remains invariant for this kind of unitary transformations.

Chiral transformations

Recall free Dirac equation: $(i\partial\!\!\!/ - m)\,\psi = 0$

and choose chiral representations for Dirac matrices

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix},\ \gamma_5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

Then Dirac equation can be rewritten as a set of two interconnected equations

$$(i\partial_t - i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\psi_L - m\psi_R = 0, \qquad (i\partial_t + i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\psi_R - m\psi_L = 0,$$

where a four component bispinor has been decomposed into two two component Weyl spinors

$$\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

Note that: $\psi_R \equiv \left(\frac{1+\gamma^5}{2}\right)\psi \quad , \quad \psi_L \equiv \left(\frac{1-\gamma^5}{2}\right)\psi$

For massless fermions (or very small masses)
left and right components are independent: chiral symmetry

Chiral transformations

Consider $\mathrm{U}(x) = e^{i\alpha(x)\gamma^5 t}$

and recall properies of γ^5

$$(\gamma^5)^2 = 1\,,$$
$$\gamma^{5\,\dagger} = \gamma^5\,,$$
$$\{\gamma^5, \gamma^0\} = 0$$

which imply $\gamma^0 \mathrm{U}^\dagger(x)\gamma^0 = \gamma^0 e^{-i\alpha(x)\gamma^5 t}\gamma^0 = e^{i\alpha(x)\gamma^5 t} = \mathrm{U}(x)$

$$\overline{\mathcal{U}} = \mathcal{U}$$
$$\det\mathcal{U} = \det\overline{\mathcal{U}}$$

and the integration measure is not invariant:

$$[\mathrm{D}\psi\mathrm{D}\overline{\psi}] \to \frac{1}{(\det\mathcal{U})^2}\,[\mathrm{D}\psi\mathrm{D}\overline{\psi}]$$

This leads to chiral anomaly, as discussed previously within the framework of perturbation theory.

Chiral anomaly

We need to calculate $\frac{1}{(\det \mathcal{U})^2}$ for $\mathrm{U}(x) = e^{i\alpha(x)\gamma^5 t}$

Consider infintensimal transformation

$$(\mathcal{U} - 1)_{xm,yn} = i\,\alpha(x)(\gamma^5 t)_{mn}\,\delta(x - y)$$

and use

$$(\det \mathcal{U})^{-2} = e^{-2\,\mathrm{tr}\,\ln \mathcal{U}}$$

$$\det \mathcal{U} = \prod_i \lambda_i = \exp\left(\sum_i \ln \lambda_i\right) = e^{\mathrm{tr}\,\ln \mathcal{U}}$$

This is very handy formula, since we can expand easily a logarithm for small U

$$\begin{aligned}(\det \mathcal{U})^{-2} &= \exp\left[-2\,\mathrm{tr}\,\ln\left(1 + i\alpha(x)\,\gamma^5\,t\,\delta(x-y)\right)\right] \\ &\underset{\alpha \ll 1}{\approx} \exp\left[-2\,i\,\mathrm{tr}\,\left(\alpha(x)\,\gamma^5\,t\,\delta(x-y)\right)\right] \\ &= \exp\left[i\int d^4x\;\alpha(x)\mathcal{A}(x)\right]\,,\end{aligned}$$

Note that trace is both for Dirac indices and for fermion species (t) and space-time.
In the last step we have introduced anomaly function, which is poorly fefined

$$\mathcal{A}(x) \equiv -2\,\mathrm{tr}\,(\gamma^5 t)\,\delta(x - x)$$

Diggression

Consider matrix

$$\mathcal{A}_{\alpha a x,\beta b y} = A_{\alpha a,\beta b}\delta^{(4)}(x-y)$$

with indices

$$\alpha,\beta - \text{ spinor}$$
$$a,b - \text{ flavor}$$
$$x,y - \text{ space-time}$$

Then

$$\begin{aligned}\mathrm{tr}\,\mathcal{A} &= \sum_{\alpha,\beta}\sum_{a,b}\int d^4x d^4y \mathcal{A}_{\alpha a x,\beta b y}\,\delta_{\alpha\beta}\delta_{ab}\delta^{(4)}(x-y) \\ &= \sum_{\alpha}\sum_{a}\int d^4x \mathcal{A}_{\alpha a x,\alpha a x} = \int d^4x\,\mathrm{tr}(A)\,\delta^{(4)}(x-x)\end{aligned}$$

Chiral anomaly

Change of integration measure under chiral transformation

$$[D\psi D\overline{\psi}] \to e^{i\int d^4x\, \alpha(x)\mathcal{A}(x)}\, [D\psi D\overline{\psi}]$$

where

$$\mathcal{A}(x) \equiv -2\,\mathrm{tr}\,(\gamma^5 t)\,\delta(x-x)$$

Note: tr gives zero and δ gives infinity.
We need to properly define this by some regularization. Before doing that, let's incorporate anomaly into the lagrangian (under functional integral):

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha(x)\mathcal{A}(x)$$

This looks like the lagrangian itself was not invariant under chiral transformation.