QCD
problem set 13

1. In QCD infinitensimal change of the gauge field under the gauge transformation
$$\Omega(x) = \exp(i\theta_a(x)T^a)$$
can be calculated from
$$\boldsymbol{A}_\mu^\Omega = \Omega^\dagger(x)\boldsymbol{A}_\mu\Omega(x) + \frac{i}{g}\Omega^\dagger(x)\partial_\mu\Omega(x)$$
and reads (show it):
$$g\,\delta A_\mu^a = g f^{abc}\theta_b(x)A_\mu^c - \partial_\mu\theta_a(x).$$
Calculate the change of the gauge condition
$$G^a(A_\mu) = n^\mu A_\mu^a$$
$g\,\delta G^a$ under this transformation.

2. Next calculate matrix
$$\mathcal{M}_{ab} = g\frac{\delta G^a}{\delta\theta_b}$$
and then rewrite the free part of the ghost action
$$\int d^4x \mathcal{L}_{FPG}^0 = \int d^4x \bar{\chi}_a \mathcal{M}_{ab}(g=0)\chi_b$$
in momentum space. The inverse of this term is the ghost propagator.

3. Applying gauge condition from problem 1 to the free part of the gluon action calculate the gluon propagator (inverse of gluonic operator derived from the free part of $F_{\mu\nu}F^{\mu\nu}$) in the axial gauge.

4. For a system of SU(3) scalar fields $\hat{\phi}_i(x)$ with $i = 1, 2, 3$ that satisfy the following commutation rules
$$\left[\hat{\phi}_i(t,\vec{x}), \hat{\pi}_j(t,\vec{x}\,')\right] = i\delta^{(3)}(\vec{x}-\vec{x}\,')\delta_{ij}$$
one defines charge operators
$$\hat{Q}^a(t) = -i\int d^3\vec{x}\,\hat{\pi}_i(t,\vec{x})T_{ij}^a\hat{\phi}_j(t,\vec{x})$$
where matrices T^a satisfy SU(3) commutation relations:
$$\left[T^a, T^b\right] = if^{abc}T^c.$$
Prove that
$$\left[\hat{Q}^a(t), \hat{Q}^b(t)\right] = if^{abc}\hat{Q}^c(t).$$

5. Flavor SU(3) left and right currents read

$$L_\mu^a = \bar{q}_L \gamma_\mu \frac{\lambda_a}{2} q_L, \; R_\mu^a = \bar{q}_R \gamma_\mu \frac{\lambda_a}{2} q_R$$

where

$$q_{\substack{L\\R}} = \frac{1}{2}\left(1 \mp \gamma_5\right) q.$$

Show that vector and axial currents take the following form

$$\begin{aligned} V_\mu^a &= R_\mu^a + L_\mu^a = \bar{q}\gamma_\mu \frac{\lambda_a}{2} q, \\ A_\mu^a &= R_\mu^a - L_\mu^a = \bar{q}\gamma_\mu \gamma_5 \frac{\lambda_a}{2} q. \end{aligned}$$

6. In the case of fermion fields, commutation relations of scalar fields are replaced by anticommutation relations:

$$\left\{ q_{\alpha,k}(t,\vec{x}), q_{\beta,l}^\dagger(t,\vec{x}\,') \right\} = \delta^{(3)}(\vec{x}-\vec{x}\,')\delta_{\alpha\beta}\delta_{kl}$$

where α, β stand for Dirac indices and k, l denote SU(3) indices. Relevant charges are defined as

$$\begin{aligned} \hat{Q}_{L,R}^a(t) &= \int d^3\vec{x}\, q_{L,R}^\dagger(t,\vec{x}) T^a q_{L,R}(t,\vec{x}), \\ \hat{Q}_V(t) &= \int d^3\vec{x} \left[q_L^\dagger(t,\vec{x}) q_L(t,\vec{x}) + q_R^\dagger(t,\vec{x}) q_R(t,\vec{x}) \right] \end{aligned}$$

where $T^a = \lambda^a/a$ are SU(3) generators (Gell-Mann matrices). Making use of the identity (prove it!)

$$\{ab, cd\} = a\{b,c\}d - ac\{b,d\} + \{a,c\}bd - c\{a,d\}b$$

show that

$$\begin{aligned} \left[\hat{Q}_L^a, \hat{Q}_L^b\right] &= i f^{abc} \hat{Q}_L^c, \\ \left[\hat{Q}_R^a, \hat{Q}_R^b\right] &= i f^{abc} \hat{Q}_R^c, \\ \left[\hat{Q}_L^a, \hat{Q}_R^b\right] &= 0, \\ \left[\hat{Q}_{L,R}^a, \hat{Q}_V\right] &= 0. \end{aligned}$$