QCD
problem set 12

1. Consider a complex scalar and/or fermion field theory coupled to the nonabelian gauge fields:
$$\begin{aligned}\mathcal{L}_\phi &= \left(D_\mu\phi(z)\right)^\dagger\left(D_\mu\phi(x)\right) - m^2\phi^\dagger(x)\phi(x) - V(\phi^\dagger(x)\phi(x)),\\ \mathcal{L}_\psi &= \bar\psi(x)\left(i\not{D}_x - m\right)\psi(x),\end{aligned}$$
with covariant derivative defined as
$$D_\mu = \partial_\mu - igA_\mu^a(x)\,T^a$$
where T^a are generators of SU(N) group. Lagrangian for the gauge fields reads
$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}^a(x)F^{a\,\mu\nu}(x).$$
Derive by means of the variational approach equations of motion for (analogs of Maxwell equations) the scalar and fermion theory. Prove the identity
$$[D_\mu, F^{\nu\rho}] + [D_\nu, F^{\rho\mu}] + [D_\rho, F^{\mu\nu}] = 0.$$

2. Show that in electrodynamics one can write
$$-\frac{1}{4}\int d^4x\, F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\int d^4x\, A^\mu(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu.$$
Transform this expression to the momentum space.

3. Generating functional for electrodynamics in a covariant gauge is defined as:
$$\begin{aligned}Z_0[j^\mu] &= \int[D\omega]\int[DA^\mu]\exp\left(-i\frac{\xi}{2}\int d^4x\,\omega^2\right)\\ &\quad\times\delta(\partial_\mu A^\mu - \omega)\exp\left(i\int d^4x\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + j_\mu A^\mu\right)\right),\end{aligned}$$
where ω, A^μ and $F^{\mu\nu}$ are functions of x. Show that it is equal to
$$Z_0[j^\mu] = \int[DA^\mu]\,\exp\left(i\int d^4x\left(\frac{1}{2}A^\mu(g_{\mu\nu}\Box - (1-\xi)\partial_\mu\partial_\nu\right)A^\nu + j_\mu A^\mu\right).$$

4. Invert matrix
$$D_{\mu\nu} = g_{\mu\nu}k^2 - (1-\xi)k_\mu k_\nu$$
looking for an inverse in the following form:
$$\left(D^{-1}\right)^{\nu\rho} = \alpha g^{\nu\rho} + \beta\frac{k^\nu k^\rho}{k^2},$$
where
$$D_{\mu\nu}\left(D^{-1}\right)^{\nu\rho} = \delta_\mu^\rho.$$
Show that there is no inverse $\left(D^{-1}\right)^{\nu\rho}$ for $\xi = 0$.