QCD

problem set 10

1. Prove that
$$\partial_\mu K^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$$
where
$$K^\mu = \varepsilon^{\mu\nu\rho\sigma}\left(A^a_\nu F^a_{\rho\sigma} - \frac{g}{3} f^{abc} A^a_\nu A^b_\rho A^c_\sigma\right).$$
This calculation proves that anomaly is a total derivative.

2. Winding number of the SU(2) gauge transformation U is defined as
$$N_{\rm w} = \frac{1}{24\pi^2}\varepsilon^{ijk}\int d^3r \; {\rm Tr}\left[\left(U^\dagger\partial_i U\right)\left(U^\dagger\partial_j U\right)\left(U^\dagger\partial_k U\right)\right]. \qquad (1)$$
Calculate (1) for $U = \exp\left(i\,\vec{n}\cdot\vec{\tau}\,P(r)\right)$ where $\vec{n} = \vec{r}/r$ (so called *hedgehog*). What are the boundary conditions for $P(r)$ that ensure that $N_{\rm w}$ is an integer?

   HINT:

   First decompose
$$U^\dagger\partial_i U = \frac{i}{2}\sum_{a=1}^{3}\xi^a_i \tau_a$$
where τ_a are Pauli matrices. You should obtain that $\varepsilon^{ijk}\,{\rm Tr}\left[\left(U^\dagger\partial_i U\right)\left(U^\dagger\partial_j U\right)\left(U^\dagger\partial_k U\right)\right] \sim \det(\xi)$.

   Due to the symmetry of U, elements of matrix ξ can be decomposed in the following way:
$$\xi^a_i = A\delta_{ia} + Bn_i n_a + C\varepsilon_{iak} n_k.$$
Express $\det(\xi)$ in terms of A, B and C. You should get an answer, which is proportional to $(A^2 + C^2)(A + B)$.

   In the last step calculate A, B and C. To this end expand U using de'Moivre (or Euler) formula for the exponent. For this you have to use that $(\vec{n}\cdot\vec{\tau})^2 = 1$.

   In order to differentiate U it is useful to use the following identities (prove them!)
$$\begin{aligned}\partial_i r &= n_i,\\ \partial_i r_k &= \frac{1}{r}(\delta_{ik} - n_i n_k).\end{aligned}$$

3. Calculate the field tensor $F_{\mu\nu}$ for the pure gauge $A_\mu = -\frac{i}{g}\Omega\partial_\mu\Omega^\dagger$ where Ω is an SU(N) transformation matrix.