QCD
problem set 9

1. Derive the Van Vleck formula

$$F(T) = \left(-\frac{1}{2\pi i\hbar} \frac{\partial^2 S}{\partial x \partial x_0} \right)^{1/2}$$

for the "quantum" part of the propagator

$$K = F(T) \exp \frac{i}{\hbar} S$$

(here $S = S_{\mathrm{cl}}$) for one dimensional problem of a particle moving in potential V. Start from the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} K(x, x_0; t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] K(x, x_0; t),$$

write

$$K = \exp \left(\frac{i}{\hbar} S + \ln F + \ldots \right)$$

and expand K in powers of $\hbar$. Show that in the first two orders in $\hbar$ the Schrödinger equation reduces to

$$\partial_t S + \frac{1}{2m} (\partial_x S)^2 + V(x) = 0, \tag{1}$$

and

$$\partial_t (\ln F) + \frac{1}{2m} \partial_x^2 S + \frac{1}{m} \partial_x S \partial_x (\ln F) = 0. \tag{2}$$

Differenciate (1) $\frac{\partial^2}{\partial x \partial x_0}$ and show that the equation obtained that way is identical to (2) where $\ln F$ has been replaced by $\frac{1}{2} \ln \frac{\partial^2 S}{\partial x \partial x_0}$ up to a constant that can be fixed from the normalization condition of the propagator for $t \to 0$:

$$K(x, x_0; t = 0) = \delta(x - x_0).$$

Compute F for the harmonic oscillator and compare with the previously obtained result.

2. Consider Gaussian integral

$$J(\mathcal{M}) = \int d^N \xi \, d^N \psi \, \exp \left(\psi_i \mathcal{M}_{ij} \xi_j \right)$$

where ψ_i and ξ_i $(i = 1, 2, \ldots N)$ are independent Grassmann variables. Expanding in a power series and commuting ξ's and ψ's show that

$$J(\mathcal{M}) = \det(\mathcal{M}).$$

3. General fermionic mass term reads (where M is complex):

$$M\,\bar{\psi}\frac{1+\gamma_5}{2}\psi + M^*\,\bar{\psi}\frac{1-\gamma_5}{2}\psi. \tag{3}$$

   Prove that (3) is Hermitean. Show that chiral transformation

$$\psi \to e^{i\alpha\gamma_5}\psi$$

   ammounts to

$$M \to e^{2i\alpha}M.$$

4. Prove that

$$\partial_\mu K^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma}$$

   where

$$K^\mu = \varepsilon^{\mu\nu\rho\sigma}\left(A^a_\nu F^a_{\rho\sigma} - \frac{g}{3}f^{abc}A^a_\nu A^b_\rho A^c_\sigma\right).$$

   This calculation proves that anomaly is a total derivative.