QCD
problem set 7 (1-5) and 8 (6-7)

1. Hopf integral.

   Calculate Hopf integral

$$I = \int_{-\infty}^{+\infty} dx \, e^{iax^2}, \; a > 0$$

   as a contour integral over the complex plane. Choose the contour in such a way that the integral $\int dt \, e^{-bt^2}$ with positive b and real t appears. When the contribution of the large circle can be neglected? Is the phase of the result unique?

2. Using the result from the previous problem calculate the following matrix element:

$$\langle y | \exp\left(-i\frac{\hat{p}^2}{2m}\epsilon\right) | x \rangle$$

   where ϵ is a small time lapse. Note that for free particle ($V = 0$) $\epsilon = T$ need not to be small and the above matrix element is in fact the exact propagator K_{free}.

3. Path integral representation for the propagator $K(b, a)$ can be derived from the Schrödinger equation, as shown at the lecture. However one can invert the logic and *postulate* the path integral representation for K. In this case one has to derive the the Schrödinger equation. To this end show that the wave function given by

$$\psi(x_2, t_2) = \int_{-\infty}^{\infty} K(x_2, t_2; x_1, t_1)\psi(x_1, t_1) \, dx_1, \tag{1}$$

   where

$$K(b, a) = \lim_{\epsilon \to 0} \int dx_1 \ldots dx_{N-1} \left(\frac{m}{2i\epsilon\hbar\pi}\right)^{N/2} e^{\frac{i\epsilon}{\hbar}\sum_{j=0}^{N-1} L_{j\to j+1}}$$

   satisfies the Schrödinger equation. Consider propagation by one "jump" in ϵ, for the case where $t_2 = t_1 + \epsilon$ and $x_1 = x_2 - \eta$. Discuss the relation between ϵ and η and expand the r.h.s. of (1) in these parameters with accuracy $\mathcal{O}(\epsilon)$(Feynman Hibbs Chapter 4-1).

4. Lagrange function for the harmonic oscillator reads:

$$L = \frac{m}{2}\dot{x}(t)^2 - \frac{m\omega^2}{2}x(t)^2.$$

   Calculate the classical trajectory leading from point $(x_a, t_a) \to (x_b, t_b)$. Calculate the classical action along this trajectory.

   HINT: After finding the classical trajectory $\bar{x}(t)$, calculate the action integrating by parts and using equations of motion.

5. For certain values $\omega(t_b - t_a) = \omega T$ both classical trajectory and classical action exhibit singularities. Find conditions that make them both finite. Discuss meaning of these conditions.

6. Show that quantum contribution to K denoted by F, where

$$K = F(T)\, e^{\frac{i}{\hbar}S[\bar{x}(t)]}$$

reads as follows

$$F(t_b - t_a) = \int [\mathcal{D}y(t)]\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} \frac{1}{2}m(\dot{y}^2 - \omega^2 y^2)\, dt} .$$

Note that the system does not distinguish any specific time, hence the amplitude may depend only on the difference $T = t_b - t_a$.

One of the methods of calculating of calculating F is to expand

$$y(t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi t}{T}\,, \qquad\qquad n > 0\,.$$

This representation of $y(t)$ satisfies the boundary conditions, $y(0) = y(T) = 0$. Note that

$$\int [\mathcal{D}y(t)] \sim \prod_n da_n$$

with all kinds of factors in front, *but we do not need to calculate them.* This is so because we know the normalization of F in the limit $\omega \to 0$, which is just the free particle propagator from problem 2. Using the fact that functions $\sin\frac{n\pi t}{T}$ form a complete set of orthogonal functions over the time interval $0 \le t \le T$ one can easily compute the argument of the exponent in F, and then perform the Gaussian integrals over da_n's. Final answer can be obtained by means of the following identity (prove it!):

$$\lim_{N\to\infty} \prod_{n=1}^{N} \left(1 - \frac{\omega^2 T^2}{n^2\pi^2}\right)^{-\frac{1}{2}} = \left(\frac{\sin\omega T}{\omega T}\right)^{-\frac{1}{2}} .$$

7. Derive the Van Vleck formula

$$F(T) = \left(-\frac{1}{2\pi i\hbar}\frac{\partial^2 S}{\partial x \partial x_0}\right)^{1/2}$$

for the "quantum" part of the propagator

$$K = F(T) \exp\frac{i}{\hbar}S$$

(here $S = S_{\rm cl}$) for one dimensional problem of a particle moving in potential V. Start from the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}K(x, x_0; t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right] K(x, x_0; t),$$

write

$$K = \exp\left(\frac{i}{\hbar}S + \ln F + \dots\right)$$

and expand K in powers of $\hbar$. Show that in the first two orders in $\hbar$ the Schrödinger equation reduces to

$$\partial_t S + \frac{1}{2m}(\partial_x S)^2 + V(x) = 0, \tag{2}$$

and

$$\partial_t(\ln F) + \frac{1}{2m}\partial_x^2 S + \frac{1}{m}\partial_x S \partial_x(\ln F) = 0. \tag{3}$$

Differenciate (2) $\frac{\partial^2}{\partial x \partial x_0}$ and show that the equation obtained that way is identical to (3) where $\ln F$ has been replaced by $\frac{1}{2}\ln\frac{\partial^2 S}{\partial x \partial x_0}$ up to a constant that can be fixed from the normalization condition of the propagator for $t \to 0$:

$$K(x, x_0; t = 0) = \delta(x - x_0).$$

Compute F for the harmonic oscillator and compare with the previous problem.