QCD
problem set 3

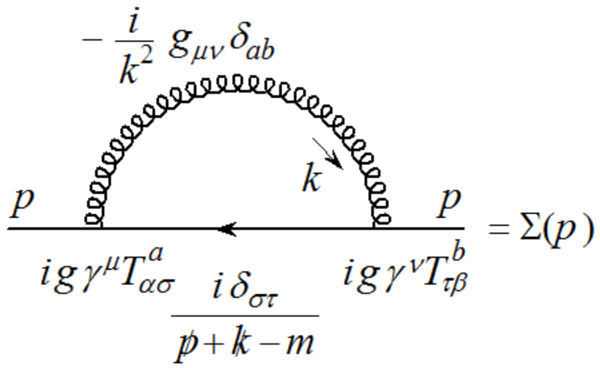

Figure 1: Feynman diagram corresponding to the quark self-energy. Time flow right to left.

1. Last time we have been computing the diagram of Fig 1, which we expressed in terms of two integrals

$$\{I, I^\mu\} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p+k)^2\, k^2} \{1, k^\mu\}\,. \tag{1}$$

for which we obtained

$$\{I, I^\mu\} = i\frac{1}{2^4\pi^2}\left(\frac{4\pi}{-p^2}\right)^\varepsilon \left\{\frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}, -p^\mu \frac{\Gamma(2-\varepsilon)}{\Gamma(3-2\varepsilon)}\right\}\Gamma(1-\varepsilon)\frac{\Gamma(\varepsilon)}{\Gamma(2)}. \tag{2}$$

Using

$$z\Gamma(z) = \Gamma(z+1), \tag{3}$$

$$\Gamma(1/2) = \sqrt{\pi}. \tag{4}$$

and

$$\Gamma(1-\varepsilon) = \exp\left(\gamma\varepsilon + \frac{\pi^2}{12}\varepsilon^2 + \dots\right) \tag{5}$$

where γ is Euler constant, show that the final answer for $\Sigma(p)$ reads

$$\Sigma(p) = i\not p\, C_F \frac{\alpha_s}{4\pi}\left(\frac{\mu^2 4\pi e^{-\gamma}}{-p^2}\right)^\varepsilon \left(\frac{1}{\varepsilon} + 1\right) \tag{6}$$

where

$$\alpha_s = \frac{g^2}{4\pi}. \tag{7}$$

2. In this problem we will calculate the d-dimensional angular integral. To this end we shall introduce spherical coordinates in d dimensions. First we chose arbitrarily a d-th axis (equivalent of the z axis in three dimensions) and project on it $\vec{k}$ vector with $\cos\theta_{d-1}$. Therefore a projection on the $d-1$ dimensional subspace orthogonal do the d-th axis is $k\sin\theta_{d-1}$. Now we choose an axis in this $d-1$ dimensional subspace, the $d-1$ axis, and project on this axis this projection (*i.e.* $k\sin\theta_{d-1}$) with $\cos\theta_{d-2}$. Next, a projection on the the $d-2$ dimensional subspace orthogonal do the d-th and $d-1$ axes involves $\sin\theta_{d-2}$. We continue this procedure until we "run out of dimensions" with the result:

$$
\begin{aligned}
k_d &= k\cos\theta_{d-1}, \\
k_{d-1} &= k\sin\theta_{d-1}\cos\theta_{d-2}, \\
&\dots \\
k_2 &= k\sin\theta_{d-1}\sin\theta_{d-2}\dots\cos\theta_1, \\
k_1 &= k\sin\theta_{d-1}\sin\theta_{d-2}\dots\sin\theta_1,
\end{aligned} \tag{8}
$$

where $\theta_1 \in (0, 2\pi), \quad \theta_{i>1} \in (0,\pi)$. Compute

$$
\int d\Omega_d = \int \prod_{i=1}^{d-1}\left(\sin^{i-1}\theta_i d\theta_i\right) = 2\prod_{i=1}^{d-1}\left(\int_0^\pi \sin^{i-1}\theta_i d\theta_i\right) \tag{9}
$$

using

$$
\int_0^\pi \sin^n\theta\, d\theta = B\left(\frac{1+n}{2}, \frac{1}{2}\right) . \tag{10}
$$

3. In this problem we will discuss parton properties assuming very simple models for quark distributions. My advice is to use *Mathematica* for these calculations.

   (a) Using properties of parton distributions given at the last page of lecture 3 http://th-www.if.uj.edu.pl/m̃ichal/QCD_2021/lecture_3.pdf

   $$
   u_v(x) = \frac{2A}{\sqrt{x}}(1-x)^3,\ d_v(x) = \frac{A}{\sqrt{x}}(1-x)^3
   $$

   where index v stands for *valence*. Recall that total u or d quark distribution is given as a sum of valence quarks and sea quarks u_s or d_s respectively. We assume that sea quark distributions are equal to antiquark distributions

   $$
   u_s(x) = \bar{u}(x),\ d_s(x) = \bar{d}(x).
   $$

   For this problem we assume that there are no strange quarks in the nucleon and we assume isospin symmetry, which says that u and d distributions in neutron, are equal to d and u distributions in proton. Calculate A. Check the value of charge of the proton and neutron. Calculate total momentum carried by the valence quarks. At this point we do not need any information on the sea quarks.

(b) Gottfried sum rule. Calculate the difference of the structure functions of the proton and neutron:

$$S_G = \int_0^1 \frac{dx}{x} \left(F_2^p(x) - F_2^n(x) \right).$$

Experimental value reads $S_G \simeq 0.24$. As you will see S_G will depend on the integral over the distributions of the sea quarks. Assume the sea quark distribution of the following form

$$\bar{u}(x) = \frac{B}{x}(1-x)^8, \ \bar{d}(x) = \frac{B}{x}(1-x)^\beta.$$

Note that constant B must be the same in both cases to assure that S_G is finite. From the experimental value of S_G calculate B for several choices of power β taking a few values around 8. Note that the antiquark and sea distributions must be positive. For these choices calculate total (valence + sea) momentum carried by quarks. Is it possible to get the value of 100%?