QCD

problem set 2

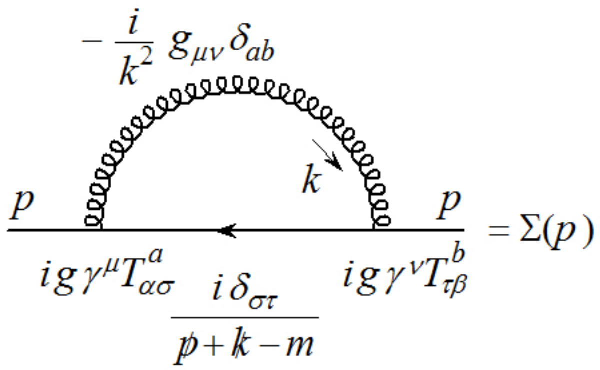

Figure 1: Feynman diagram corresponding to the quark self-energy. Time flow right to left.

In this problem set we shall perform detailed calculation of the fermion self-energy in QCD depicted in Fig. 1. Assume $p^2 \neq 0$. Note that the diagram (1) is almost the same as in QED, except for the color T generators, that enter in the quark-gluon vertices. Gluon propagator is in the Feynman gauge. The problem is divided into a few steps.

1. Write mathematical expression $\Sigma(p)$ corresponding to the diagram (1). Show that the only effect of the fact that we calculate this diagram in QCD is a *color factor* in front.

2. In the loop diagram (1) change the 4-dimensional integral over the gluon momentum k to a d dimensional one according to the following prescription:

$$\frac{d^4k}{(2\pi)^d} \to \mu^{4-d}\frac{d^dk}{(2\pi)^d}$$

   Convince yourself that if we assume that

$$4 \to d = 4 - 2\varepsilon$$

   where $\varepsilon \to 0_+$ then the integral over k is finite. The method of changing dimensionality of space-time, known as *dimensional regularization* proposed by Veltman and 't Hooft, has great advantage over some other regularization methods, namely it preserves gauge invariance. Note that in order to preserve dimensionality of $\Sigma(p)$ we had to introduce an arbitrary mass parameter μ.

3. In the following we shall put $m = 0$. Using

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

and

$$g_{\mu\nu} g^{\mu\nu} = d.$$

calculate the numerator of $\Sigma(p)$. You should obtain that

$$\Sigma(p) \quad \sim \quad \int \frac{d^d k}{(2\pi)^d} \frac{\not{p} + \not{k}}{(p+k)^2\, k^2} = \not{p} I + \gamma_\mu I^\mu$$

4. To calculate integrals

$$\{I, I^\mu\} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p+k)^2\, k^2} \{1, k^\mu\}\,. \tag{1}$$

introduce now Feynman parametrization for the propagators in (1), change variables

$$k^\mu \to k^\mu + x p^\mu \tag{2}$$

and introduce

$$M^2 = -x(1-x)\, p^2\,. \tag{3}$$

You should arrive at:

$$\{I, I^\mu\} = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{(\,k^2 - M^2)^2} \{1, k^\mu - x p^\mu\}\,. \tag{4}$$

5. In order to calculate the integral over $d^d k$, which is the integral in Minkowski space, we observe that (to see this reinstall Feynman $+i\epsilon$ prescription):

$$\left\{ \int_{-\infty}^{\infty} + \int_{C_R} + \int_{+i\infty}^{-i\infty} \right\} dk^0 = 0. \tag{5}$$

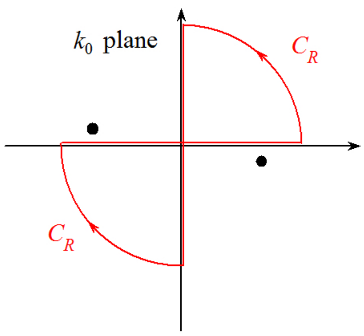

Figure 2: Integration contour over k_0. Black dots denote poles of Feynman propagators.

Since the integral over C_R vanishes

$$\int_{-\infty}^{\infty} dk^0 = -\int_{+i\infty}^{-i\infty} dk^0 = i \int_{-\infty}^{+\infty} dE \tag{6}$$

where $k^0 = iE$. Therefore the integral over $d^d k$ in Minkowski space transforms into the Euclidean integral

$$\{I, I^\mu\} = i \int_0^1 dx \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{\left(-\vec{k}^2 - M^2\right)^2} \{1, k^\mu - xp^\mu\} \tag{7}$$

where

$$\vec{k} = (E, k^1, k^2, \ldots, k^{d-1}). \tag{8}$$

6. Since nothing depends on the angles, except of k^μ, which is nullified by the angular integration, we can use (we shall prove this later, but because full angular integral corresponds to the surface of a sphere of radius $r = 1$ in d dimensions, you can check that the formula below is right for $d = 2$ or 3):

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{9}$$

After angular integration we arrive at (using $d = 4 - 2\epsilon$):

$$\{I, I^\mu\} = \frac{i}{\Gamma(2-\varepsilon)} \frac{2\pi^{2-\varepsilon}}{(2\pi)^{4-2\varepsilon}} \int_0^1 dx \, \{1, -xp^\mu\} \int_0^\infty dk \frac{k^{d-1}}{(k^2 + M^2)^2}. \tag{10}$$

Changing variables to $r = k/M$, and then $t = r^2$, you should get two integrals that are representations of the Euler beta functions:

$$\begin{aligned} \int_0^\infty dt \, \frac{t^{x-1}}{(1+t)^{x+y}} &= B(x, y), \\ \int_0^1 dx \, x^{\alpha-1}(1-x)^{\beta-1} &= B(\alpha, \beta). \end{aligned} \tag{11}$$

Identify values of x, y, β and α and then write the final expression for I and I^μ in terms of Euler Γ functions only (use the well known expression for beta functions).