QCD lecture 9

December 9

Quantization of QCD

In QED gauge fixing resulted in an infinte constant that could be discarded.

We have decomposed the gauger field into two components: $A^\mu = A^\mu_\perp + A^\mu_\parallel$

getting $[DA^\mu] = [DA^\mu_\perp]\,[DA^\mu_\parallel]$

$$
\begin{aligned}
Z_0[j^\mu] \equiv & \int [DA^\mu_\parallel(x)]\ \exp\left\{i \int d^4x\, j_\mu A^\mu_\parallel\right\} \\
& \times \int [DA^\mu_\perp(x)]\ \exp\left\{i \int d^4x\, \left(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + j_\mu A^\mu_\perp\right)\right\}
\end{aligned}
$$

Recall $\widetilde{A}^\mu_\parallel(k) \equiv \left(\frac{k^\mu k^\nu}{k^2}\right)\widetilde{A}_\nu(k)$ but vector current is conserved $k^\mu j_\mu = 0$

and $\int [DA^\mu_\parallel(x)]$ is an infinite constant that has to be divided out.

This is no longer true in QCD.

Quantization of QCD

Consider expectation value of some gauge invariant operator

$$\langle \mathcal{O} \rangle \equiv \int [DA^a_\mu(x)]\ \mathcal{O}(A_\mu)\ \exp\left\{ i \underbrace{\int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} \right)}_{\mathcal{S}_{YM}[A_\mu]} \right\}$$

When we perform gauge transformation

$$A_\mu(x) \quad \to \quad A^\Omega_\mu(x) \equiv \Omega^\dagger(x)\, A_\mu(x)\, \Omega(x) + \frac{i}{g}\, \Omega^\dagger(x)\, \partial_\mu \Omega(x)$$

the integration measure changes

$$[DA^\Omega_{a\,\mu}(x)] = [DA_{a\,\mu}(x)]\ \det\left[\left(\frac{\delta A^\Omega_{a\,\mu}(x)}{\delta A_{b\,\nu}(y)} \right) \right]$$

We need to calculate the Jacobian. For this we need a small reminder from group theory.

Diggression

Consider some representation r

$$\boldsymbol{X} = X_a T^a(r), \ \boldsymbol{Y} = Y_a T^a(r)$$

Let's calculate an object analogous to gauge transformation:

$$\begin{aligned} e^{-i\boldsymbol{X}}\boldsymbol{Y}e^{+i\boldsymbol{X}} &= \boldsymbol{Y} - i\left[\boldsymbol{X},\boldsymbol{Y}\right] + \ldots \\ &= \boldsymbol{Y} - iX_aY_b\left[T^a,T^b\right] + \ldots \\ &= \left[Y_c - i(-if_{acb})X_aY_b\right]T^c + \ldots \\ &= \left[Y_c - iX_a\left(T^a_{\text{adj}}\right)_{cb}Y_b + \ldots\right]T^c \end{aligned}$$

This can be written in short $\left[e^{-i\boldsymbol{X}}\boldsymbol{Y}e^{+i\boldsymbol{X}}\right]_c = \left[\exp\left(-iX_aT^a_{\text{adj}}\right)\right]_{cb}Y_b$

Gauge transformation $\Omega^\dagger\boldsymbol{A}\Omega = e^{-i\text{ad}_\Omega}\boldsymbol{A}$

$$\frac{\delta A^\Omega_{a\mu}(x)}{\delta A_{b\nu}(y)} = \delta^\nu_\mu\delta(x-y)\left(e^{-i\text{ad}_\Omega}\right)_{ab} \Longrightarrow \det\left(\frac{\delta A^\Omega_{a\mu}(x)}{\delta A_{b\nu}(y)}\right) = 1$$

Quantization of QCD

Changing gauge does not change the integration measure

$$[DA^{\Omega}_{a\,\mu}(x)] = [DA_{a\,\mu}(x)]$$

So the path integral is infinite. To eliminate gauge redundancy we have to fix the gauge.

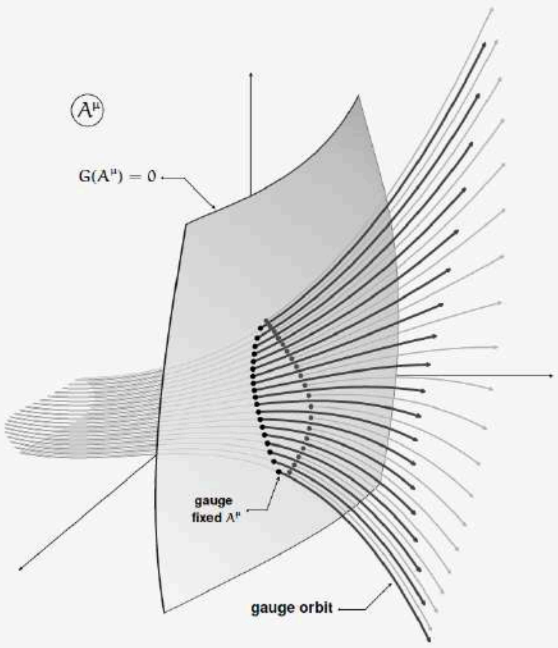


Figure 5.1: Illustration of the gauge fixing procedure. The lines represent the gauge field configurations spanned when varying Ω. The shaded surface is the manifold where the gauge condition is satisfied, and the black dots are the gauge-fixed field configurations.

$$G^a(A_\mu(x)) = 0$$

[This condition may have many solutions (Gribov copies) but only one of them is perturbative, others are $\sim 1/g$]

We want to split the functional integration into a physical component in the gauge fixing manifold and a component along the gauge orbit (analogue of transverse QED field). This can be done by inserting

$$\delta[G^a(A_\mu)]$$

into the functional integral.

How this behaves under the gauge transformation?

Quantization of QCD

Toy model example $\quad f(x_0) = 0$

$$\int dx\,\delta(f(x)) = \int dx \frac{1}{|f'(x)|}\delta(x - x_0) = \frac{1}{|f'(x)|}\bigg|_{x=x_0}$$

Define $\quad \Delta^{-1}[A_\mu] \equiv \int [D\Omega(x)]\ \delta[G^a(A_\mu^\Omega)]$

then

$$\Delta(A_\mu) = \det\left(\frac{\delta G^a}{\delta\Omega}\right)_{G^a(A_\mu^\Omega)=0}$$

[Faddeev – Popov determinant]

In QED Δ does not depend on A_μ but in QCD it does, becuse gauge tranformation is non-linear:

$$A_\mu^\Omega(x) \equiv \underline{\Omega^\dagger(x)\,A_\mu(x)\,\Omega(x)} + \frac{i}{g}\,\Omega^\dagger(x)\,\partial_\mu\Omega(x)$$

Quantization of QCD

First we prove that $\Delta[A_\mu]$ is gauge invariant

$$
\begin{aligned}
\Delta^{-1}[A_\mu^\Theta] &= \int [D\Omega(x)]\, \delta[G^a(A_\mu^{\overbrace{\Theta\Omega}^{\Omega'}})] \\
&= \int [D(\Theta^\dagger(x)\Omega'(x))]\, \delta[G^a(A_\mu^{\Omega'})] \\
&= \int [D\Omega'(x)]\, \delta[G^a(A_\mu^{\Omega'})] = \Delta^{-1}[A_\mu]
\end{aligned}
$$

Last step follows from the unitarity of gauge transformations (there exists a group invariant measure on a Lie group).

Hence

$$
1 = \Delta[A_\mu] \int [D\Omega(x)]\, \delta[G^a(A_\mu^\Omega)]
$$

and we will insert this unity under the functional integral.

Quantization of QCD

Expectation value of gauge invariant operator:

$$\langle \mathcal{O} \rangle = \int \underline{[D\Omega(x)]} \int [DA_\mu^a(x)]\ \Delta[A_\mu]\ \delta[\underline{G^a(A_\mu^\Omega)}]\ \mathcal{O}(A_\mu)\ e^{i\mathcal{S}_{YM}[A_\mu]}$$

Change variables: $A_\mu \rightarrow A_\mu^{\Omega^\dagger}$

Invariants:

$$\begin{aligned} [DA_\mu^{\Omega^\dagger}] &= [DA_\mu]\,, \\ \mathcal{S}_{YM}[A_\mu^{\Omega^\dagger}] &= \mathcal{S}_{YM}[A_\mu]\,, \\ \mathcal{O}[A_\mu^{\Omega^\dagger}] &= \mathcal{O}[A_\mu]\,, \\ \Delta[A_\mu^{\Omega^\dagger}] &= \Delta[A_\mu]\,, \end{aligned}$$

At this point the functional integral does not contain the gauge transformation

$$\langle \mathcal{O} \rangle = \int \underline{[D\Omega(x)]} \int [DA_\mu^a(x)]\ \Delta[A_\mu]\ \delta[G^a(A_\mu)]\ \mathcal{O}(A_\mu)\ e^{i\mathcal{S}_{YM}[A_\mu]}$$

We can now drop $[D\Omega]$. So functional integral has been factored out into a gauge orbit part at the expense of $\Delta[A_\mu]$ that modifies QCD Feynman rules.

Quantization of QCD

We need to find a functional representation for the Faddeev-Popov determinant. Recall (lecture 5)

$$\det(\mathbf{M}) \equiv \int d^N\xi d^N\psi \ \exp(\psi_i M_{ij} \xi_j)$$

Let's introduce new fermion fields (Faddeev-Popov ghosts)

$$\det(i\mathcal{M}) = \int [D\chi_a(x) D\overline{\chi}_a(x)] \\ \times \exp\left\{ i \int d^4x d^4y \, \overline{\chi}_a(x) \, \mathcal{M}_{ab}(x,y) \, \chi_b(y) \right\}$$

and use a trick for covariant gauges in QED (lecture 5)

$$\int [D\omega(x)] \ \exp\left\{ -i\frac{\xi}{2} \int d^4x \ \omega^2(x) \right\} \ \delta[G^a(A_\mu(x)) - \omega]$$

Quantization of QCD

After integration over $D[\omega]$

$$\begin{aligned}\langle \mathcal{O} \rangle &= \int [DA^a_\mu(x)]\,[D\chi_a(x) D\overline{\chi}_a(x)]\;\mathcal{O}(A_\mu) \\ &\times \exp i \int d^4x \left(\underbrace{-\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}}_{\mathcal{L}_{YM}} \underbrace{-\frac{\xi}{2} (G^a(A_\mu))^2}_{\mathcal{L}_{GF}} + \underbrace{\overline{\chi}_a\, \mathcal{M}_{ab}\, \chi_b}_{\mathcal{L}_{FPG}} \right)\end{aligned}$$

Note that $\mathcal{M}_{ab} \sim g \left(\frac{\delta G^a}{\delta \Omega_b} \right)$ at $\Omega = 1$ (det is gauge inv.)

and therefore is a function of A_μ. Ghost fields couple to the gauge fields and appear only inside loops. In practice they remove contributions from the "longitudinal" gauge fields. They ensure that the theory is unitary.

Both GF and FPG depend on the gauge choice (choice of function G). Typically we choose G linear in A_μ, so the gluon propagator will depend on ξ and will be the same as in QED, up to the color factor.

[Matrix $\mathcal{M}_{ab}$ can be scaled by any factor $\mathcal{M} \to \kappa \mathcal{M}$, this changes the propagator $S \to \kappa^{-1} S$ and vertices $V \to \kappa V$ leaving the final result invariant.]

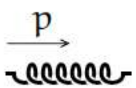
Covariant gauge

EXAMPLE

Covariant gauge $\quad G^a(A) \equiv \partial^\mu A^a_\mu - \omega^a(x)$

gluon propagator (as in QED)

$$G^{0\,\mu\nu}_{F\,ab}(p) =$$



$$= \frac{-i\,g^{\mu\nu}\,\delta_{ab}}{p^2 + i0+} + \frac{i\,\delta_{ab}}{p^2 + i0+}\left(1 - \frac{1}{\xi}\right)\frac{p^\mu p^\nu}{p^2}$$

We need to calculate matrix $\mathcal{M}_{ab}$

Gauge transformation $\quad A^\Omega_\mu(x) \equiv \Omega^\dagger(x)\,A_\mu(x)\,\Omega(x) + \frac{i}{g}\,\Omega^\dagger(x)\,\partial_\mu\Omega(x) \qquad \Omega(x) = \exp(i\theta_a(x)T^a)$

infinitensimal $\quad g\,\delta A_{a\,\mu}(x) = g\,f^{abc}\,\theta_b(x)\,A_{c\,\mu}(x) - \partial_\mu\theta_a(x)$

which yields: $\quad g\;\delta G^a = g\,f^{abc}\left(\partial^\mu\theta_b(x)\right)A_{c\,\mu}(x) + g\,f^{abc}\,\theta_b(x)\left(\partial^\mu A_{c\,\mu}(x)\right) - \Box\,\theta_a(x)$

and:

$$\mathcal{M}_{ab} = g\,\frac{\delta G^a(A)}{\delta\theta^b} = g\,f^{abc}\left(\partial^\mu A_{c\,\mu}(x)\right) + g\,f^{abc}\,A_{c\,\mu}(x)\,\partial^\mu - \delta_{ab}\,\Box$$

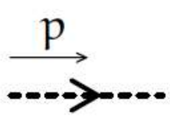
So this matrix contains gluon-ghost interactions and ghost propagator (inverse)

Covariant gauge

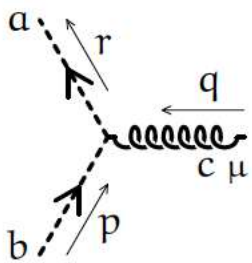
This results in the following FPG langrangian:

$$\mathcal{L}_{FPG} = \overline{\chi}_a \left(-\delta_{ab}\,\Box + g\,f^{abc}\,(\partial^\mu A_{c\,\mu}(x)) + g\,f^{abc}\,A_{c\,\mu}(x)\,\partial^\mu \right) \chi_b$$

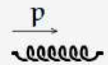
and the following Feynman rules:

$G^0_F(p) =$ 

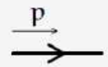
$$= \frac{i\,\delta_{ab}}{p^2 + i0^+}$$



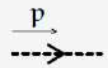
$$= g\,f^{abc}\,(p_\mu + q_\mu) = g\,f^{abc}\,r_\mu$$



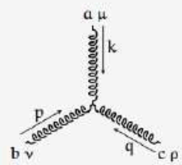
$$= \frac{-i\, g^{\mu\nu}\, \delta_{ab}}{p^2 + i0^+} + \frac{i\, \delta_{ab}}{p^2 + i0^+} \left(1 - \frac{1}{\xi}\right) \frac{p^\mu p^\nu}{p^2}$$



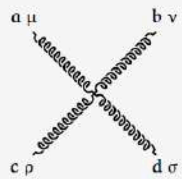
$$= \frac{i\, \delta_{ij}}{\not{p} - m + i0^+}$$



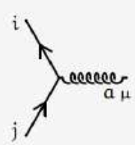
$$= \frac{i\, \delta_{ab}}{p^2 + i0^+}$$



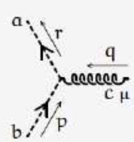
$$= g\, f^{abc} \left\{ g^{\mu\nu} (k - p)^\rho + g^{\nu\rho} (p - q)^\mu + g^{\rho\mu} (q - k)^\nu \right\}$$



$$\begin{aligned} = -i\, g^2 \big\{ & f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) \big\} \end{aligned}$$



$$= -i\, g\, \gamma^\mu \left(t_r^a\right)_{ij}$$



$$= g\, f^{abc} (p_\mu + q_\mu) = g\, f^{abc}\, r_\mu$$

Figure 5.2: Feynman rules of non-Abelian gauge theories in covariant gauge. We also list the rules involving fermions for completeness. Latin characters a, b, c refer to the adjoint representation, while the letters i, j refer to the representation r in which the fermions live.

Axial gauge

Usefull class of gauges

$$G^a(A) \equiv n^\mu A^a_\mu - \omega^a(x)$$

where n^μ is a fixed four-vector. If it is time-like – temporal gauge
light-like – light-cone gauge

Then (exercise)

$$\frac{1}{2} A^a_\mu \left(g^{\mu\nu}\Box - \partial^\mu\partial^\nu - \xi n^\mu n^\nu\right) A^a_\nu$$

and we have to invert the following matrix:

$$g^{\mu\nu}p^2 - p^\mu p^\nu + \xi n^\mu n^\nu$$

which gives (exercise):

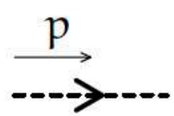
$$G^{0\,\mu\nu}_{F\,ab}(p) = \frac{-i\delta_{ab}}{p^2 + i0^+}\left[g^{\mu\nu} - \frac{p^\mu n^\nu + p^\nu n^\mu}{p \cdot n} + \frac{p^\mu p^\nu}{(p \cdot n)^2}\left(n^2 + \xi^{-1}p^2\right)\right]$$

Axial gauge

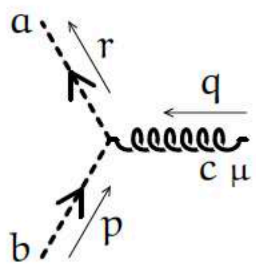
Final result (exercise)

$$\mathcal{L}_{FPG} = \overline{\chi}_a \left(-\delta_{ab}\, n^\mu \partial_\mu + g\, f^{abc}\, n^\mu A_{c\,\mu}(x) \right) \chi_b$$

and the ghost propagator and ghost vertex look like:

$G_F^0(p) =$ 

$= -\dfrac{\delta_{ab}}{p \cdot n + i0^+}$



$= i\, g\, f^{abc}\, n_\mu \,.$

Chiral symmetry

Quark masses (from sum rules at μ = 1 GeV)

$$\begin{pmatrix} m_u = 0.005\,\mathrm{GeV} \\ m_d = 0.009\,\mathrm{GeV} \\ m_s = 0.175\,\mathrm{GeV} \end{pmatrix} \ll 1\,\mathrm{GeV} \leq \begin{pmatrix} m_c = (1.15 - 1.35)\,\mathrm{GeV} \\ m_b = (4.0 - 4.4)\,\mathrm{GeV} \\ m_t = 174\,\mathrm{GeV} \end{pmatrix}$$

Approximate symmetry: up, down, strange are massless.

QCD lagrangian ($\mathcal{G}_a^{\mu\nu}$ - field tensor)

$$\mathcal{L}^0_{\mathrm{QCD}} = \sum_{l=u,d,s} \bar{q}_l i \not{D}\, q_l - \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}_a^{\mu\nu}$$

We know, that right-handed and left-hanfded fermions transform independently:

$$P_R = \frac{1}{2}(1+\gamma_5) = P_R^\dagger, \quad P_L = \frac{1}{2}(1-\gamma_5) = P_L^\dagger \qquad P_R + P_L = 1,$$

Chiral symmetry

Define

$$q_R = P_R q, \quad q_L = P_L q$$

and rewrite the lagrangian

$$\mathcal{L}^0_{\mathrm{QCD}} = \sum_{l=u,d,s} \left(\bar{q}_{R,l} i \not{D}\, q_{R,l} + \bar{q}_{L,l} i \not{D}\, q_{L,l} \right) - \frac{1}{4} \mathcal{G}_{\mu\nu,a} \mathcal{G}^{\mu\nu}_a$$

Chiral symmetry (global $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$)

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i \sum_{a=1}^{8} \Theta^L_a \frac{\lambda_a}{2} \right) e^{-i\Theta^L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}$$

$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp\left(-i \sum_{a=1}^{8} \Theta^R_a \frac{\lambda_a}{2} \right) e^{-i\Theta^R} \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

Parity

Consider Dirac equation

$$\left\{i\gamma^0\partial_t + i\boldsymbol{\gamma}\cdot\partial_{\mathbf{x}} - m\right\}\psi(t,\mathbf{x}) = 0$$

and space reflection $\mathbf{x} \to -\mathbf{x}$

then $$\left\{i\gamma^0\partial_t - i\boldsymbol{\gamma}\cdot\partial_{\mathbf{x}} - m\right\}\psi(t,-\mathbf{x}) = 0$$

What is the wave function transformation generated by space reflection?
We have to change sign of space gamma matrices and leave unchanges time gamm matrix:

$$\gamma^0 = P^{-1}\gamma^0 P \qquad\qquad -\boldsymbol{\gamma} = P^{-1}\boldsymbol{\gamma} P$$

Then $$\left\{i\gamma^0\partial_t + i\boldsymbol{\gamma}\cdot\partial_{\mathbf{x}} - m\right\}\psi^P(t,\mathbf{x}) = 0$$

where

$$\psi^P(t,\mathbf{x}) = P\,\psi(t,-\mathbf{x})$$

Parity

We need to solve $\gamma^0 = P^{-1}\gamma^0 P$ $\quad\quad -\gamma = P^{-1}\gamma P$

and the solution reads (exercise): $P = P^{-1} = \gamma^0$

Parity transformation $P : q(\vec{x}, t) \mapsto \gamma_0 q(-\vec{x}, t)$

changes chirality (because γ^0 anticommutes with γ^5)

$$q_R(\vec{x}, t) = P_R\, q(\vec{x}, t) \mapsto P_R \gamma_0 q(-\vec{x}, t) = \gamma_0 P_L q(-\vec{x}, t) \neq \pm q_R(-\vec{x}, t)$$

Parity transforms left and right fermions into each other.

QCD physical states (mesons) should be grouped in multiplets of some representations of $\mathrm{U}(3)_L \times \mathrm{U}(3)_R$ and, because of the fact that parity transformation changes chiralty, multiplets with positive and negative parity should be gegenrate (in mass). This is not observed experimentally. We will make this statement more precise later.

Conserved currents

Recall Noether theorem:
in order to find conserved currents of some global symmetry transformation, we have to promote this symmetry to a local one and calculate the currents.

Consider $$\mathcal{L} = \mathcal{L}(\Phi_i, \partial_\mu \Phi_i)$$

which leads to the equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} = 0, \quad i = 1, \cdots, n$$

Suppose fields $\Phi_i(x)$ transform according to some symmtry group (local). Consider infinitensimal transformation

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) + \delta\Phi_i(x) = \Phi_i(x) - i\epsilon_a(x) F_i^a[\Phi_j(x)]$$

which is not necessarily linear

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) - i\epsilon_a(x) t_{ij}^a \Phi_j(x)$$

Conserved currents

Field transformation

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) + \delta\Phi_i(x) = \Phi_i(x) - i\epsilon_a(x) F_i^a[\Phi_j(x)]$$

Variation of the lagrangian

$$\begin{aligned} \delta\mathcal{L} &= \mathcal{L}(\Phi_i', \partial_\mu \Phi_i') - \mathcal{L}(\Phi_i, \partial_\mu \Phi_i) \\ &= \frac{\partial \mathcal{L}}{\partial \Phi_i} \delta\Phi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \partial_\mu \delta\Phi_i \\ &= \epsilon_a(x) \left(-i \frac{\partial \mathcal{L}}{\partial \Phi_i} F_i^a - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \partial_\mu F_i^a \right) + \partial_\mu \epsilon_a(x) \left(-i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} F_i^a \right) \end{aligned}$$

Define current $\quad J^{\mu,a} = -i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} F_i^a$

and calculate
its divergence
and use EoM

$$\begin{aligned} \partial_\mu J^{\mu,a} &= -i \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \right) F_i^a - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \partial_\mu F_i^a \\ &= -i \frac{\partial \mathcal{L}}{\partial \Phi_i} F_i^a - i \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} \partial_\mu F_i^a , \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi_i} = 0$$

Conserved currents

We arrive at $\delta\mathcal{L} = \epsilon_a(x)\partial_\mu J^{\mu,a} + \partial_\mu\epsilon_a(x) J^{\mu,a}$

This allows do define currents and current derivarives as

$$
\begin{aligned}
J^{\mu,a} &= \frac{\partial\delta\mathcal{L}}{\partial\partial_\mu\epsilon_a}, \\
\partial_\mu J^{\mu,a} &= \frac{\partial\delta\mathcal{L}}{\partial\epsilon_a}.
\end{aligned}
$$

If we demand the action to be invariant, we can integrate last term by parts, and we conclude that the current is conserved:

$$\partial_\mu J^{\mu,a} = 0$$

It follows that there exists a consrved charge (exercise)

$$Q^a(t) = \int d^3x J_0^a(\vec{x}, t)$$

Currents in QFT

Canonical quantization
define generalized momenta $\Pi_i = \partial\mathcal{L}/\partial(\partial_0\Phi_i)$

and impose commutation rules:

$$\begin{aligned}
[\Phi_i(\vec{x},t), \Pi_j(\vec{y},t)] &= i\delta^3(\vec{x}-\vec{y})\delta_{ij}, \\
[\Phi_i(\vec{x},t), \Phi_j(\vec{y},t)] &= 0, \\
[\Pi_i(\vec{x},t), \Pi_j(\vec{y},t)] &= 0.
\end{aligned}$$

Suppose now that the symmetry transformation is linear

$$\Phi_i(x) \mapsto \Phi_i'(x) = \Phi_i(x) - i\epsilon_a(x) t^a_{ij}\Phi_j(x)$$

then (current and charge are operators now, normal ordering suppressed)

$$\begin{aligned}
J^{\mu,a}(x) &= -it^a_{ij}\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\Phi_j, \\
Q^a(t) &= -i\int d^3x\, \Pi_i(x) t^a_{ij}\Phi_j(x)
\end{aligned}$$

Commutation rules

It is easy to show (exercise)

$$
\begin{aligned}
[Q^a(t), \Phi_k(\vec{y}, t)] &= -it^a_{ij} \int d^3x \, [\Pi_i(\vec{x}, t)\Phi_j(\vec{x}, t), \Phi_k(\vec{y}, t)] \\
&= -t^a_{kj}\Phi_j(\vec{y}, t).
\end{aligned}
$$

Field (operator) transformations induce transformations of the Hilbert space

$$|\alpha'\rangle = [1 + i\epsilon_a G^a(t)]|\alpha\rangle$$

where G^a are hermitian operators (they in principle could depend on time).
We demand

$$\langle\beta|A|\alpha\rangle = \langle\beta'|A'|\alpha'\rangle$$

Commutation rules

For a matrix element of a filed we have

$$\begin{aligned}\langle\beta|\Phi_i(x)|\alpha\rangle &= \langle\beta'|\Phi_i'(x)|\alpha'\rangle \\ &= \langle\beta|[1-i\epsilon_a G^a(t)][\Phi_i(x)-i\epsilon_b t^b_{ij}\Phi_j(x)][1+i\epsilon_c G^c(t)]|\alpha\rangle\end{aligned}$$

terms linear in ε should vanish

$$0 = -i\epsilon_a[G^a(t),\Phi_i(x)] \underbrace{-i\epsilon_a t^a_{ij}\Phi_j(x)}_{i\epsilon_a[Q^a(t),\Phi_i(x)]} ,$$

From this we conclude that $G^a(t) = Q^a(t)$

Commutation rules

Finally

$$[Q^a(t), Q^b(t)] \;=\; -i(t^a_{ij}t^b_{jk} - t^b_{ij}t^a_{jk}) \int d^3x\, \Pi_i(\vec{x},t)\Phi_k(\vec{x},t)$$

Recalling that

$$t^a_{ij}t^b_{jk} - t^b_{ij}t^a_{jk} = iC_{abc}t^c_{ik}$$

We have

$$[Q^a(t), Q^b(t)] = iC_{abc}Q^c(t)$$