QCD lecture 8

December 2

Euclidean path integral

There is no classical trajectory: $-a \to a$ Go to Euclidean time $t = -i\tau$

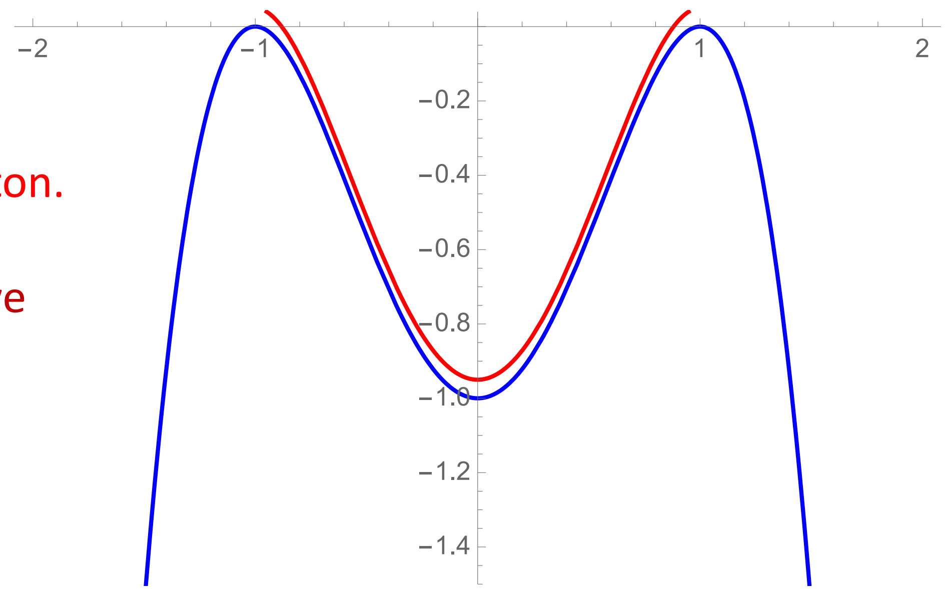
where

$$K_E(x_b, \tfrac{1}{2}T; x_a, -\tfrac{1}{2}T) \;=\; <x_a|e^{-\frac{1}{\hbar}HT}|x_a> = \int [\mathcal{D}_E x(\tau)] e^{-\frac{1}{\hbar}S_E[x(\tau)]}$$

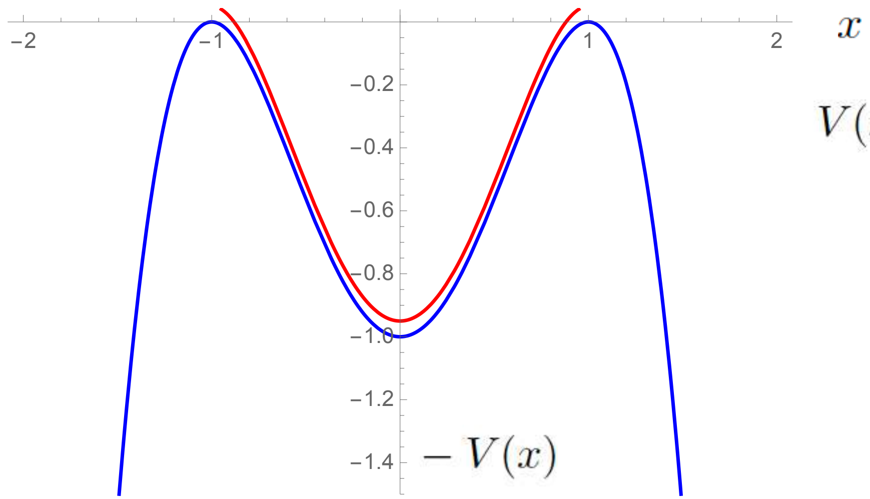
$$S_E[x(\tau)] = \int_{-T/2}^{T/2} d\tau \left[\tfrac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]$$

Potential is inverted and there is a classical trajectory called instanton.

To calculate the energy splitting we have to sum over an infinite number of instantons

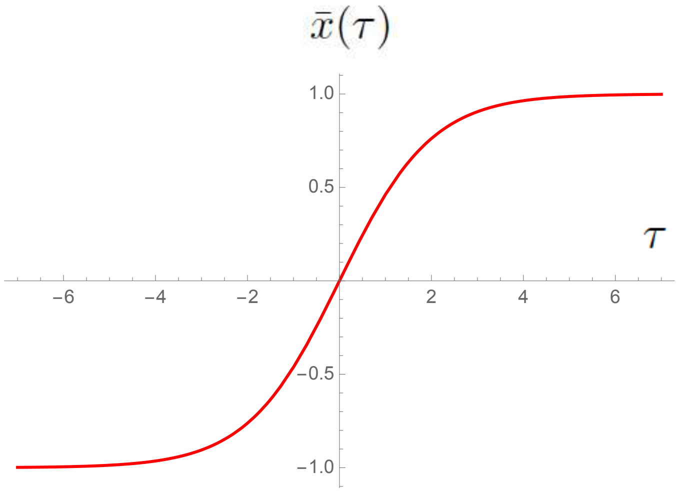


Explicit model



$$V(x) = \frac{1}{8a^2}(a^2 - x^2)^2$$

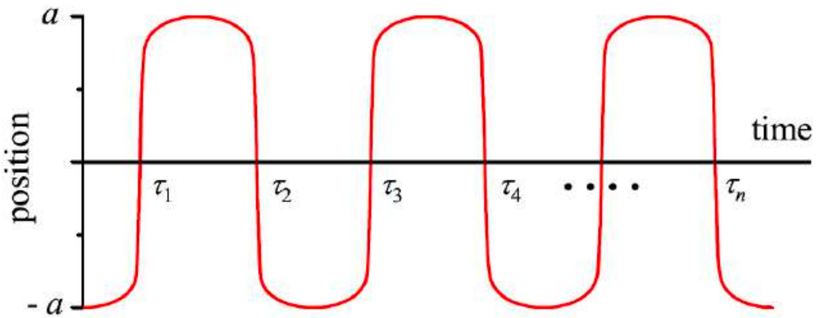
Instanton is an Euclidean trajectory of zero energy.



$$|\bar{x} - a| \sim e^{-\sqrt{\frac{V''(a)}{m}}\tau} = e^{-\omega\tau}$$

$$\bar{x}(\tau) = a \tanh \frac{\tau - \tau_1}{2}$$

Multi-instanton transition amplitude



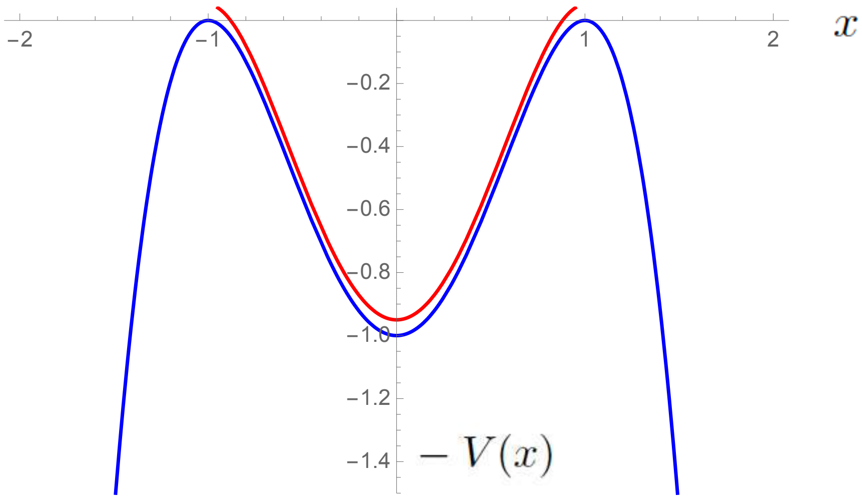
$$x(\tau) = \bar{x}_{\tau_1 \ldots \tau_n}(\tau) + y(\tau) \approx \bar{x}_{\tau_1}(\tau) + \bar{x}_{\tau_2}(\tau) + \ldots + \bar{x}_{\tau_n}(\tau) + y(\tau)$$

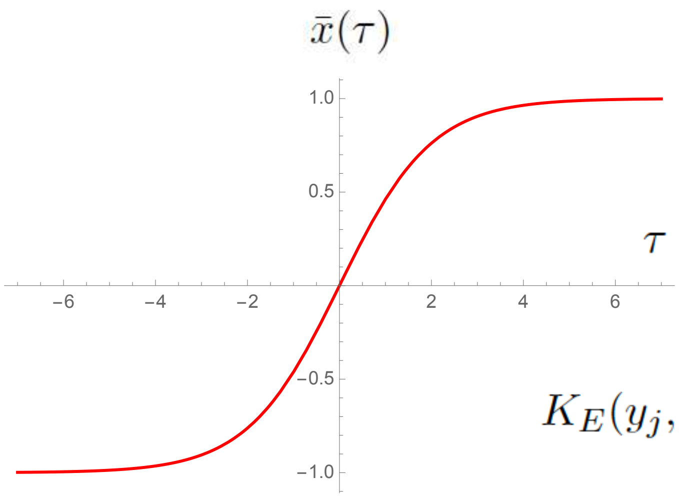
Here $\bar{x}_{\tau_1 \ldots \tau_n}(\tau)$ is the exact classical trajectory that can be approximated by a sum over one-(anti) instanton trajectories $\bar{x}_{\tau_n}$ where $\tau_1, \ldots, \tau_n$ mark times of individual jumps.

$$<-a|e^{-\frac{1}{\hbar}HT}|-a> = \sum_{\text{even}\, n} \int_{-T/2}^{T/2} d\tau_1 \ldots \int_{\tau_{n-2}}^{T/2} d\tau_{n-1} \int_{\tau_{n-1}}^{T/2} d\tau_n \, e^{-\frac{1}{\hbar} S_E[\bar{x}_{\tau_1 \ldots \tau_n}(\tau)]}$$

$$\times \int_{y(-\frac{T}{2})=y(+\frac{T}{2})=0} [\mathcal{D}y_E(\tau)] \, e^{-\frac{1}{2\hbar} \int_{-T/2}^{+T/2} d\tau y(-m\frac{d^2}{d\tau^2} + V''(\bar{x}))y}$$

Oscillator approximation





We are considering fluctuations around one instanton. But for most of the time the particle is either in one or the other maximum (minimum in Minkowski space) i.e. it sits there and does not move. This corresponds to a trivial classical trajectory of an Euclidean oscillator (potential is quadratic around each maximum). Quantal operator

$$\left(-m\frac{d^2}{d\tau^2}+V''(\bar{x})\right) \qquad \omega^2=\frac{V''(\pm a)}{m}$$

is the same in either maximum. So we can approximate fluctuations around one instanton

$$K_E(y_j,T_j;y_{j-1},T_{j-1}) = \tilde{K}\,K_E^{\rm osc}(y_j,T_j;y_{j-1},T_{j-1})$$

where $\tilde{K}$ is a correction factor.

Energy splitting

$$< -a|e^{-\frac{1}{\hbar}HT}| -a > \approx \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\omega T} \sum_{\text{even } n} \frac{1}{n!}\left(\tilde{K}e^{-S_E^0/\hbar}T\right)^n$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\omega T}\frac{1}{2}\left[e^{\tilde{K}e^{-S_E^0/\hbar}T} + e^{-\tilde{K}e^{-S_E^0/\hbar}T}\right]$$

$$= \frac{1}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \left[e^{-(\frac{1}{2}\omega - \tilde{K}e^{-S_E^0/\hbar})T} + e^{-(\frac{1}{2}\omega + \tilde{K}e^{-S_E^0/\hbar})T}\right]$$

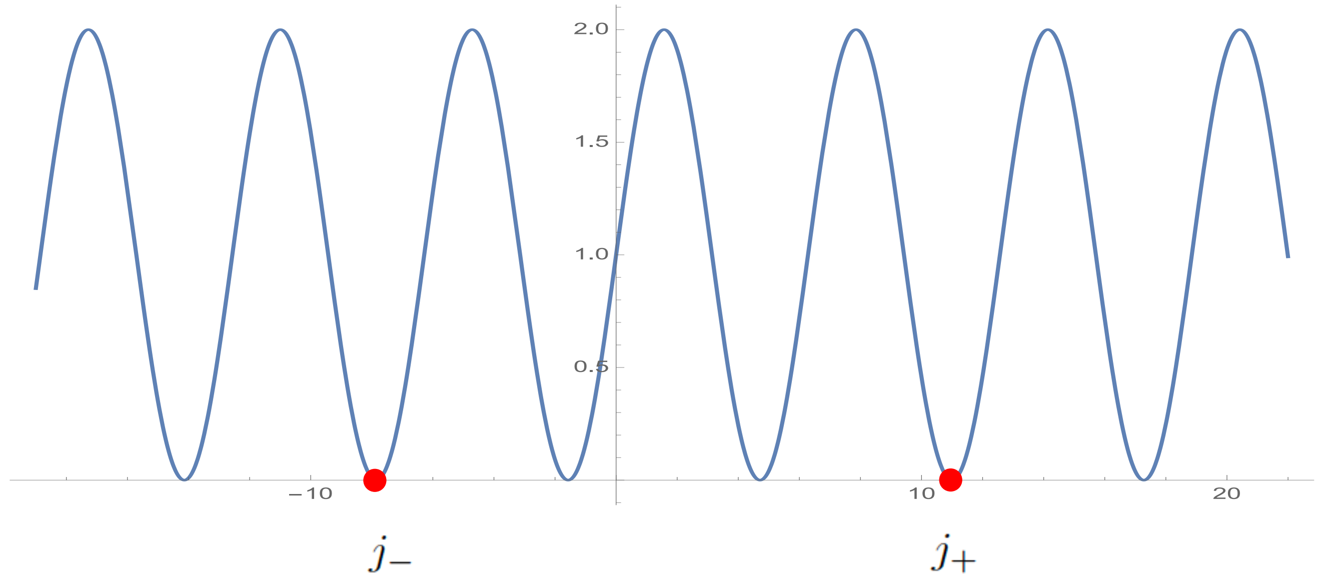
Because in this limit only the the ground state survives, we have two lowest energies

$$E_s = \tfrac{1}{2}\hbar\omega - \hbar\tilde{K}e^{-S_E^0/\hbar}$$

$$E_r = \tfrac{1}{2}\hbar\omega + \hbar\tilde{K}e^{-S_E^0/\hbar}$$

Splitting is nonperturbative suppressed by the exponent from the classical action

Periodic potential



Minima are labelled by integer $j = ... -2, -1, 0, +1, +2, ...$ $x_j(\tau) = ja$

we jump right n times and left m times.

Let's calculate the amplitude $< j_+ | e^{-\frac{1}{\hbar}HT} | j_- >$

Periodic potential

$$< j_+ | e^{-\frac{1}{\hbar} HT} | j_- > \; = \; \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{2}} e^{-\frac{1}{2}\omega T}$$

$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{n!\, m!} \left(\tilde{K} e^{-S_E^0/\hbar} T \right)^{n+m} \delta_{(n-m)(j_+ - j_-)}$$

Use the following representation

$$\delta_{(n-m)(j_+ - j_-)} = \int_0^{2\pi} \frac{d\theta}{2\pi} \, e^{i\theta(n-m)} e^{i\theta(j_- - j_+)}$$

$$< j_+ | e^{-\frac{1}{\hbar} HT} | j_- > = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{2}} e^{-\frac{1}{2}\omega T} \int_0^{2\pi} \frac{d\theta}{2\pi} \, e^{i\theta(j_- - j_+)}$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \left(\tilde{K} T e^{-S_E^0/\hbar + i\theta} \right)^n \sum_{m=0}^{\infty} \frac{1}{m!} \left(\tilde{K} T e^{-S_E^0/\hbar - i\theta} \right)^m$$

Periodic potential

$$<j_+|e^{-\frac{1}{\hbar}HT}|j_-> = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\omega T} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(j_- - j_+)} e^{\tilde{K}Te^{-S_E^0/\hbar + i\theta}} e^{\tilde{K}Te^{-S_E^0/\hbar - i\theta}}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\omega T} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(j_- - j_+)} e^{2\tilde{K}Te^{-S_E^0/\hbar}\cos\theta}$$

So the ground state band

$$\tfrac{1}{2}\hbar\omega \;\rightarrow\; E(\theta) = \tfrac{1}{2}\hbar\omega - 2\tilde{K}\hbar e^{-S_E^0/\hbar}\cos\theta$$

Calculation of $\tilde{K}$

$$K_E(y_j, T_j; y_{j-1}, T_{j-1}) = \tilde{K}\, K_E^{\rm osc}(y_j, T_j; y_{j-1}, T_{j-1})$$

Note that $\tilde{K}$ is a number given by a ratio of the square root of two determinants

$$\tilde{K} = \frac{[\det(-m\frac{d^2}{d\tau^2} + m\omega^2))]^{\frac{1}{2}}}{[\det\,(-m\frac{d^2}{d\tau^2} + V''(\bar{x}))]^{\frac{1}{2}}}$$

For the instanton we need to find eigenvalues of

$$\left(-m\frac{d^2}{d\tau^2} + \frac{d^2V}{d\bar{x}^2}\right)y_n(\tau) = \lambda_n y_n(\tau)$$

We will show now that this operator has one zero mode (!). This would in principle render $\tilde{K}$ infinite.

Instanton zero mode

Recall that instanton has zero (Euclidean) energy:

$$\tfrac{1}{2}m\dot{\bar{x}}^2 - V(x) = 0 \quad \Longrightarrow \quad \frac{d\bar{x}}{d\tau} = \left(\frac{2}{m}V(\bar{x})\right)^{\frac{1}{2}}$$

Let's differenciate velocity over time: $\displaystyle \frac{d^2\bar{x}}{d\tau^2} = \frac{d}{d\bar{x}}\left[\left(\frac{2}{m}V(\bar{x})\right)^{\frac{1}{2}}\right]\frac{d\bar{x}}{d\tau} = \frac{1}{m}\frac{dV}{d\bar{x}}$

and once more: $\displaystyle \left(-m\frac{d^2}{d\tau^2} + \frac{d^2V}{d\bar{x}^2}\right)\frac{d\bar{x}}{d\tau} = 0$

But this is our eigen-equation for a zero mode $\lambda_1 = 0$ (this the lowest eigen-value):

$$\left(-m\frac{d^2}{d\tau^2} + \frac{d^2V}{d\bar{x}^2}\right)y_n(\tau) = \lambda_n y_n(\tau)$$

We can normalize this mode (exercise)

$$y_1(\tau) = \left(S_E^0\right)^{-\frac{1}{2}}\sqrt{m}\,\frac{d\bar{x}}{d\tau} \qquad \text{where} \qquad S_E^0 = \int_{-T/2}^{+T/2} d\tau\, 2V(\bar{x})$$

Instanton zero mode

Consider one instnaton trajectory

$$x(\tau) = \bar{x}(\tau) + y(\tau) = \bar{x}(\tau) + a_1 y_1(\tau) + \sum_{l>1} a_l \, y_l(\tau) \qquad \int \prod_i da_i$$

Note that $\bar{x}(\tau) = \bar{x}_{\tau_1}(\tau) = \bar{x}(\tau - \tau_1)$

Change of the trajectory due to the change of the jump time τ_1 is equal to

$$dx(\tau) = \frac{d\bar{x}}{d\tau_1} d\tau_1 = -\frac{d\bar{x}}{d\tau} d\tau_1 = -\sqrt{\frac{S_E^0}{m}} \, y_1 \, d\tau_1$$

But this is the change corresponding to the zero mode

$$dx(\tau) = y_1 da_1$$

So we have already taken this change into accound when integrating over jump times. This is the exact result (while integrations over da_i are in Gaussian approximation). We therefore have to omit $\lambda_1 = 0$ in the instanton determinant, include Jacobian for the change of variables and remove $\sqrt{2\pi\hbar}$ arising from the Gaussian integral.

Instanton in QM: summary

$$\tilde{K} = \left(\frac{S_E^0}{m2\pi\hbar}\right)^{\frac{1}{2}} \frac{[\det(-m\frac{d^2}{d\tau^2} + m\omega^2))]^{\frac{1}{2}}}{[\det{}'(-m\frac{d^2}{d\tau^2} + V''(\bar{x}))]^{\frac{1}{2}}}$$

Here prime means: no zero mode

Instantons in Minkowski space correspond to the tunelling between the minima of the potential.

In Euclidean space instantons are *localized* (around τ_1) solutions of classical equations of motion that in infinty go to the different vacua.

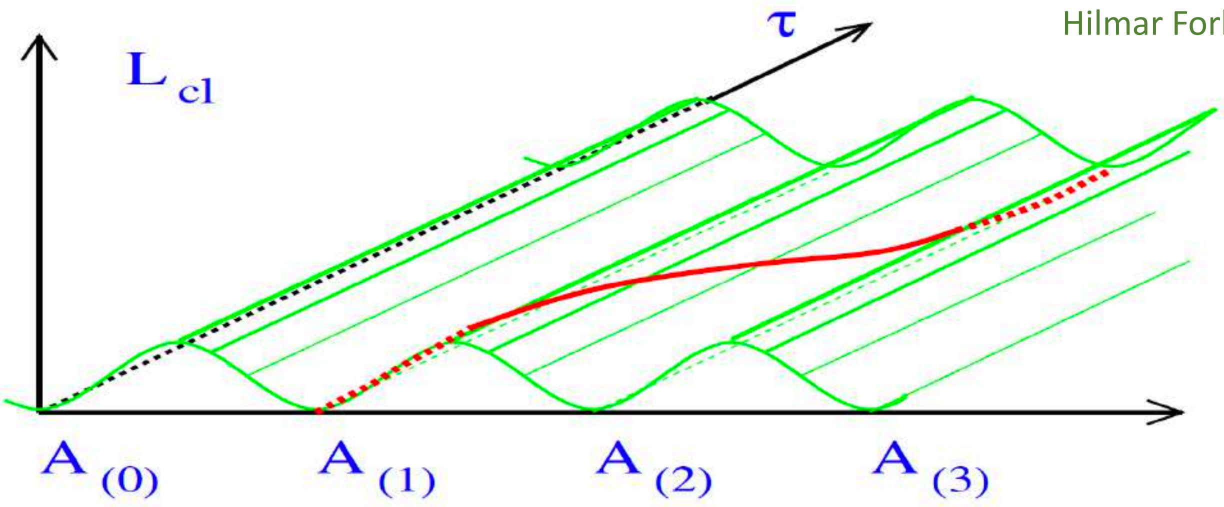
Instanton quantal operator for fluctuations around classical trajectory has a zero mode.

Zero modes have to be omitted from the quantal determinant and taken care off exactly.

Instantons give rise to the splitting of naively degenarate energy eigen-states. This splitting is non-perturbative and exponenially suppressed.

Instantons in QCD

In order to continously deform $A_\mu^{(n)} \to A_\mu^{(m)}$ we have to consider field configurations with nonminimal action $S_E > 0$



Hilmar Forkel *A Primer on Instnatons*

Instantons are solutions of the Euclidean equations of motion (QCD or Yang Mills eqs.)

$$D_\mu^{ab} F_{\mu\nu}^b = 0$$

with the following boundary conditions:

$$A_\mu(\vec{x}, T = -\infty) = A_\mu^{(n)}(\vec{x}),$$
$$A_\mu(\vec{x}, T = +\infty) = A_\mu^{(n+1)}(\vec{x})$$

They are time dep. solutions of $n = 1$ and minimal possible action.

Instantons in QCD

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Instantons in QCD

Instantons satisfy important property. Define dual field tensor $\tilde{F}^a_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^a_{\alpha\beta}$

Recall: $$\frac{g^2}{32\pi^2}\int d^4x\,\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta} = \frac{g^2}{32\pi^2}\int d^4x\,F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = N_W\ (=1)$$

$$S = \frac{1}{4}\int d^4x\,F^a_{\mu\nu}F^a_{\mu\nu}$$ Euclidean action has +sign

Construct a positive quantity:

$$0 \leq \int d^4x\,\left(F^a_{\mu\nu} \pm \tilde{F}^a_{\mu\nu}\right)^2 = \int d^4x\,\left(2F^a_{\mu\nu}F^a_{\mu\nu} \pm 2F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}\right) = 8S \pm \frac{64\pi^2}{g^2}N_W$$

which gives a Bogomolny bound

$$S \geq \frac{8\pi^2}{g^2}\,|N_W|$$

Instantons minimize the action, so they are self-dual solutions $F^a_{\mu\nu} = \tilde{F}^a_{\mu\nu}$

Explicit instanton solution

$$A_\mu^a(x) = \frac{2}{g}\eta^a_{\mu\nu}\frac{(x-z)_\nu}{(x-z)^2+\rho^2}$$

't Hooft symbols

$$\eta^a_{\mu\nu} = \begin{cases} \varepsilon^{a\mu\nu} & \mu,\nu = 1,2,3 \\ -\delta^{a\nu} & \mu = 4 \\ +\delta^{a\mu} & \nu = 4 \\ 0 & \mu=\nu=4 \end{cases}$$

change sign for anti-instantons

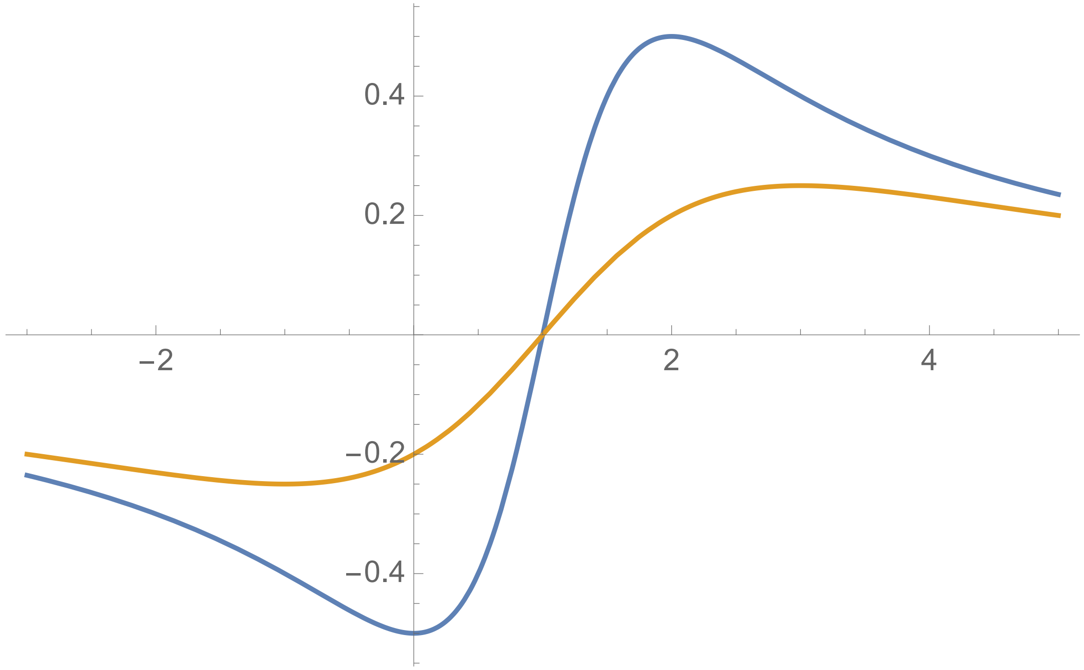
$$\eta^1_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \eta^2_{\mu\nu} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \eta^3_{\mu\nu} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

They are localized solutions. z_μ is called instanton center, ρ instanton size

Explicit instanton solution

$$A^a_\mu(x) = \frac{2}{g}\eta^a_{\mu\nu}\frac{(x-z)_\nu}{(x-z)^2+\rho^2}$$

One dim. plot:



They are localized solutions. z_μ is called instanton center, ρ instanton size

Collective coordinates

Once we have classical solution we have to calculate quantal deteminant. However, like in QM, there will be zero modes corresponding to the flat directions of the classical action (within a topological class):

- change of instnanton center z_μ - 4
- change of size ρ - 1
- 3 parameters of a global gauge transformation (or 3 rotations)

Therefore there are 8 zero modes. In QM we had one zero mode corresponding to τ_1

Consider fluctuations around the classical configuration

$$A = A^{\text{inst}} + a$$

Then

$$\begin{aligned} S[A^{\text{inst}} + a] &= \frac{8\pi^2}{g^2} + \frac{1}{2}\int d^4x \int d^4y\, a(x)\mathcal{D}(x,y)a(y) \\ &\to e^{-8\pi^2/g^2}\frac{1}{\sqrt{\det \mathcal{D}}} = e^{-8\pi^2/g^2}\prod_s \lambda_s^{-1/2} \end{aligned}$$

Collective coordinates

$$e^{-8\pi^2/g^2}\frac{1}{\sqrt{\det \mathcal{D}}} = e^{-8\pi^2/g^2}\prod_s \lambda_s^{-1/2}$$

However, for each zero mode we do not integrate over a complete set of eigen-functions of $\mathcal{D}$ but w perform an exact integration over the zero modes. Recall that there is a Jacobian between the two. In QM we had

$$\sqrt{\frac{S_E^0}{m}}\, y_1\, d\tau_1 \quad \text{instead of} \quad y_1 da_1$$

This holds also in the QCD case:

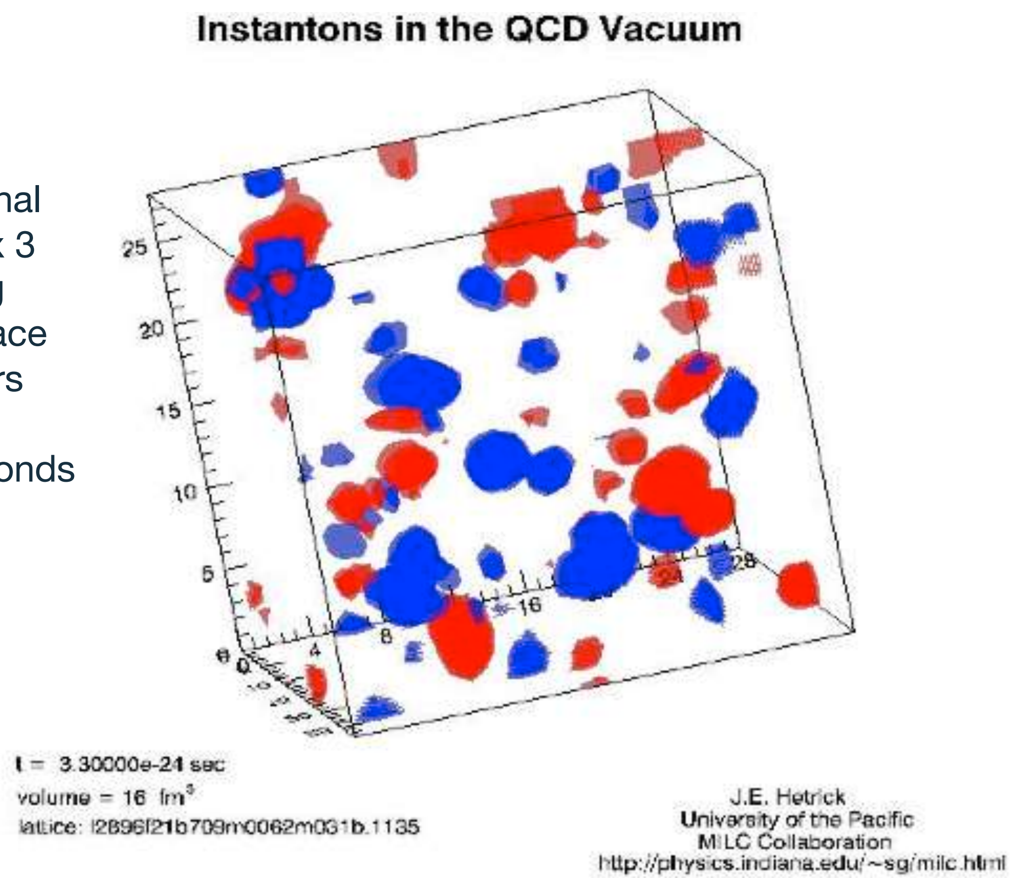
$$e^{-8\pi^2/g^2}\prod_{\substack{\text{zero}\\ \text{modes}}}\frac{1}{g}\prod_{\substack{s\neq\text{zero}\\ \text{mode}}}\lambda_s^{-1/2} \sim e^{-8\pi^2/g^2}\frac{1}{g^8}$$

This shows how highly non-perturbative are instanton contributions to the expectation values in QCD.
Nonzero modes make coupling constant running.

Lattice QCD

Each lattice is a four-dimensional array (283 x 96, say) of four 3 x 3 complex matrices representing these fields in a tiny box of space measuring about 2 femtometers on a side (1 fm = 10^{-15} m) and extending about 10^{-22} seconds in time.

Instantons and anti-instantons

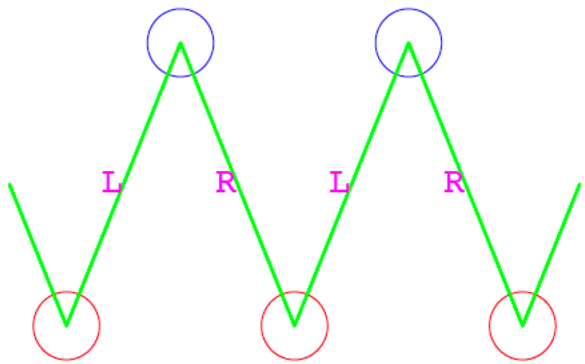


https://www.nersc.gov/news-publications/nersc-news/nersc-centernews/2004/nersc-s-qcd-library-is-a-model-resource/

Chiral symmetry breaking

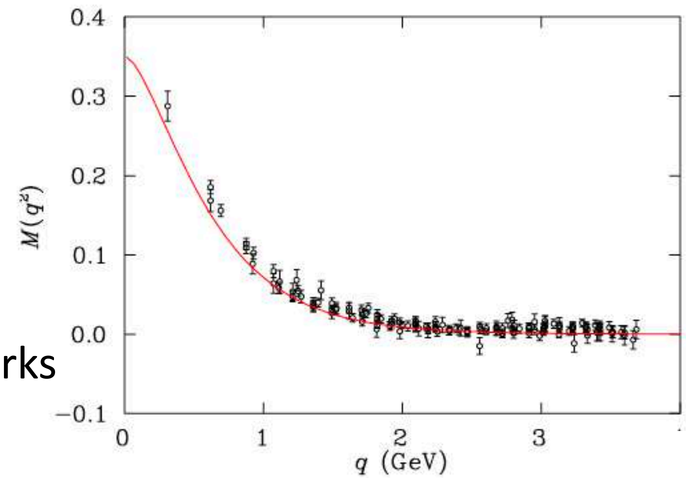
$$\frac{g^2}{32\pi^2}\int d^4x_E\ \epsilon_{ijkl}\,F^a_{ij}(x)\,F^b_{kl}(x)\,\mathrm{tr}(t^a t^b) = n_R - n_L$$

Quark propagaring between instantons and anti-instantons changes chirality.



This leads to the chiral symmetry breaking, quarks get constituent mass that is momentum dependent

This mechanism explains why proton has mass of 1 GeV, while current (Higgs) masses of u,d quarks are ~a few MeV



Average instanton zize ρ = 1/3 fm and R = 1 fm (average distance between instantons)

Diakonov 2003 *Instantons at work*

$$\epsilon_{vac} \simeq -\frac{b_1}{128\pi^2}\langle 0\,|g^2G^2|\,0\rangle \simeq -\frac{1}{2}\frac{\mathrm{GeV}}{\mathrm{fm}^3}$$

Instantons and θ term

In principle we should include the sum over all topological sectors in the QCD path integral

$$\langle \mathcal{O} \rangle = Z^{-1} \sum_{n \in \mathbb{Z}} P(n) \int [DA]_n \, \mathcal{O}[A] \, e^{-\mathcal{S}[A]}$$

where $P(n)$ is a weight factor and measure $[DA]_n$ is restricted to topological sector n. One can prove

$$P(n_1 + n_2) = P(n_1) P(n_2)$$

the solution is $P(n) = e^{-n\theta}$ where θ is an arbitrary constant. However

$$n = \frac{g^2}{64\pi^2} \int d^4x \, \epsilon^{ijkl} F^a_{ij} F^a_{kl}$$

So we may add theta term to the QCD lagrangian and integrate over all A fields. Note that $\theta = 0$ corresponds to the uniform weight factor.

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