QCD lecture 6

November 18

Fermionic symmetries

Consider a set of fermion fields $\psi(x)$ (with components $\psi_n(x)$) interacting with a gauge potential $A^a_\mu(x)$

At this moment we do not specify the meaning of index *n*

- it can be color index
- it can be flavor (up, down, strange ...)
- it can be spinor index
- or some combination of the above

What is important is the unitary transformation of these fields

$$\psi(x) \to U(x)\psi(x) \qquad \psi^\dagger(x) \to \psi^\dagger(x)U^\dagger(x)$$

Dirac conjugate: $\overline{\psi}(x) \equiv \psi^\dagger(x)\gamma^0 \to \psi^\dagger(x)U^\dagger(x)\gamma^0 = \overline{\psi}(x)\underline{\gamma^0 U^\dagger(x)\gamma^0}$

These matrices have both fermionic and space-time indices: $\overline{\mathcal{U}}$

$$\mathcal{U}_{xm,yn} \equiv U_{mn}(x)\,\delta(x-y)\,,$$
$$\overline{\mathcal{U}}_{xm,yn} \equiv (\gamma^0 U^\dagger(x)\gamma^0)_{mn}\,\delta(x-y)$$

Fermionic symmetries

EXAMPLE:

$$\mathcal{U}(x) = e^{i\alpha(x)t}$$

where $\alpha(x) \in \mathbb{R}$ and t is hermitean matrix (generator) that carries no spinor indices.

Then multiplication is understood as (remember that $(\gamma^0)^2 = 1$):

$$\begin{aligned}(\overline{\mathcal{U}}\mathcal{U})_{xm,yn} &= \int d^4z \sum_p \overline{\mathcal{U}}_{xm,zp}\, \mathcal{U}_{zp,yn} \\ &= \int d^4z\, \delta(x-z)\delta(z-y) \sum_p \left(e^{-i\alpha(z)t}\right)_{mp} \left(e^{i\alpha(z)t}\right)_{pn} \\ &= \delta_{mn}\delta(x-y)\,.\end{aligned}$$

This means that $\overline{\mathcal{U}}\mathcal{U} = 1$, which implies that $\det \mathcal{U} \det \overline{\mathcal{U}} = 1$

Since under such transformation the Grassmann integration measure changes as

$$[D\psi D\overline{\psi}] \to \frac{1}{\det(\mathcal{U}) \det(\overline{\mathcal{U}})} [D\psi D\overline{\psi}]$$

the measure remains invariant for this kind of unitary transformations.

Chiral transformations

Recall free Dirac equation: $(i\partial\!\!\!/ - m)\,\psi = 0$

and choose chiral representations for Dirac matrices

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \; \gamma_5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

Then Dirac equation can be rewritten as a set of two interconnected equations

$$(i\partial_t - i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\psi_L - m\psi_R = 0, \qquad (i\partial_t + i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\,\psi_R - m\psi_L = 0,$$

where a four component bispinor has been decomposed into two two component Weyl spinors

$$\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

Note that: $\psi_R \equiv \left(\frac{1+\gamma^5}{2}\right)\psi \quad , \quad \psi_L \equiv \left(\frac{1-\gamma^5}{2}\right)\psi$

For massless fermions (or very small masses)
left and right components are independent: chiral symmetry

Chiral transformations

Consider $\quad U(x) = e^{i\alpha(x)\gamma^5 t}$

and recall properies of γ^5

$$(\gamma^5)^2 = 1\,,$$
$$\gamma^{5\dagger} = \gamma^5\,,$$
$$\{\gamma^5, \gamma^0\} = 0$$

which imply $\quad \gamma^0 U^\dagger(x)\gamma^0 = \gamma^0 e^{-i\alpha(x)\gamma^5 t}\gamma^0 = e^{i\alpha(x)\gamma^5 t} = U(x)$

$$\overline{\mathcal{U}} = \mathcal{U}$$
$$\det\mathcal{U} = \det\overline{\mathcal{U}}$$

and the integration measure is not invariant:

$$[D\psi D\overline{\psi}] \to \frac{1}{(\det\mathcal{U})^2}\,[D\psi D\overline{\psi}]$$

This leads to chiral anomaly, as discussed previously within the framework of perturbation theory.

Chiral anomaly

We need to calculate $\frac{1}{(\det \mathcal{U})^2}$ for $U(x) = e^{i\alpha(x)\gamma^5 t}$

Consider infintensimal transformation

$$(\mathcal{U}-1)_{xm,yn} = i\,\alpha(x)(\gamma^5 t)_{mn}\,\delta(x-y)$$

and use

$$(\det \mathcal{U})^{-2} = e^{-2\,\mathrm{tr}\,\ln \mathcal{U}}$$

$$\det \mathcal{U} = \prod_i \lambda_i = \exp\left(\sum_i \ln \lambda_i\right) = e^{\mathrm{tr}\,\ln \mathcal{U}}$$

This is very handy formula, since we can expand easily a logarithm for small U

$$\begin{aligned}(\det \mathcal{U})^{-2} &= \exp\left[-2\,\mathrm{tr}\,\ln\left(1 + i\alpha(x)\,\gamma^5\,t\,\delta(x-y)\right)\right] \\ &\underset{\alpha\ll 1}{\approx} \exp\left[-2\,i\,\mathrm{tr}\,\left(\alpha(x)\,\gamma^5\,t\,\delta(x-y)\right)\right] \\ &= \exp\left[i\int d^4x\;\alpha(x)\mathcal{A}(x)\right],\end{aligned}$$

Note that trace is both for Dirac indices and for fermion species (*t*) and space-time.
In the last step we have introduced anomaly function, which is poorly fefined

$$\mathcal{A}(x) \equiv -2\,\mathrm{tr}\,(\gamma^5 t)\,\delta(x-x)$$

Diggression

Consider matrix

$$\mathcal{A}_{\alpha a x,\beta b y} = A_{\alpha a,\beta b}\delta^{(4)}(x-y)$$

with indices

$$\alpha,\beta - \text{ spinor}$$
$$a,b - \text{ flavor}$$
$$x,y - \text{ space-time}$$

Then

$$\begin{aligned}\mathrm{tr}\,\mathcal{A} &= \sum_{\alpha,\beta}\sum_{a,b}\int d^4x d^4y \mathcal{A}_{\alpha a x,\beta b y}\,\delta_{\alpha\beta}\delta_{ab}\delta^{(4)}(x-y) \\ &= \sum_{\alpha}\sum_{a}\int d^4x \mathcal{A}_{\alpha a x,\alpha a x} = \int d^4x\,\mathrm{tr}(A)\,\delta^{(4)}(x-x)\end{aligned}$$

Chiral anomaly

Change of integration measure under chiral transformation

$$[D\psi D\overline{\psi}] \to e^{i\int d^4x\ \alpha(x)\mathcal{A}(x)}\ [D\psi D\overline{\psi}]$$

where $$\mathcal{A}(x) \equiv -2\,\mathrm{tr}\,(\gamma^5 t)\,\delta(x-x)$$

Note: tr gives zero and δ gives infinity.
We need to properly define this by some regularization. Before doing that, let's incorporate anomaly into the lagrangian (under functional integral):

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha(x)\mathcal{A}(x)$$

This looks like the lagrangian itself was not invariant under chiral transformation.

Chiral anomaly in gauge theory

Assume that our fermions couple to a (non)-Abelian gauge field through covariant derivative

$$\not{D}_x \equiv \gamma^\mu \left(\partial_\mu - i\, g\, t^a A^a_\mu(x)\right)$$ (note different convetion for g)

This means that matrix t may have both flavor and color indices (typicaly it is a product of flavor and color matrix).

Fujikawa proposed the following regularization (M - regularization parameter, nothing can depend on M)

$$\mathcal{A}(x) = -2 \lim_{y \to x, M \to +\infty} \operatorname{tr} \left\{ \gamma^5\, t\, \mathcal{F}\left(-\frac{\not{D}_x^2}{M^2} \right) \right\} \delta(x-y)$$

where function $\mathcal{F}(s)$ has the following properties:

$$\mathcal{F}(0) = 1\,,$$
$$\mathcal{F}(+\infty) = 0\,,$$
$$s\,\mathcal{F}'(s) = 0 \quad \text{at } s = 0 \quad \text{and at } s = +\infty$$

Note that this regularization is gauge invarint due to the covariant derivative (as a consequence vector current is conserved) .

Chiral anomaly in gauge theory

We need to calculate
$$\mathcal{A}(x) = -2 \lim_{y\to x, M\to+\infty} \mathrm{tr}\left\{\gamma^5\, t\, \mathcal{F}\left(-\frac{\not{D}_x^2}{M^2}\right)\right\} \delta(x-y)$$

use
$$\delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}$$

to get
$$\begin{aligned} \mathcal{A}(x) &= -2 \int \frac{d^4k}{(2\pi)^4} \lim_{y\to x, M\to+\infty} \mathrm{tr}\left\{\gamma^5\, t\, \mathcal{F}\left(-\frac{\not{D}_x^2}{M^2}\right)\right\} e^{ik(x-y)} \\ &= -2 \int \frac{d^4k}{(2\pi)^4} \lim_{M\to+\infty} \mathrm{tr}\left\{\gamma^5\, t\, \mathcal{F}\left(-\frac{(i\not{k}+\not{D}_x)^2}{M^2}\right)\right\} . \end{aligned}$$

Second equality follows from:
$$\lim_{y\to x} \mathcal{F}(\partial_x)\, e^{ik\cdot(x-y)} = \mathcal{F}(ik+\partial_x)$$

Recall $\not{D}_x \equiv \gamma^\mu \left(\partial_\mu - i\, g\, t^a A_\mu^a(x)\right)$ Change integration variable $k \to Mk$

$$\mathcal{A}(x) = -2 \lim_{M\to+\infty} M^4 \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left\{\gamma^5\, t\, \mathcal{F}\left(-\left[i\not{k} + \frac{\not{D}_x}{M}\right]^2\right)\right\}$$

Chiral anomaly in gauge theory

We have $$\mathcal{A}(x) = -2 \lim_{M\to+\infty} M^4 \int \frac{d^4k}{(2\pi)^4} \,\mathrm{tr}\left\{\gamma^5 \,t\, \mathcal{F}\left(-\left[i\not{k} + \frac{\not{D}_x}{M}\right]^2\right)\right\}$$

We have to square $$-\left[i\not{k} + \frac{\not{D}_x}{M}\right]^2 = k^2 - 2i\frac{k\cdot D_x}{M} - \left(\frac{\not{D}_x}{M}\right)^2$$

We have used
$$\not{k}^2 = k_\mu k_\nu \gamma^\mu\gamma^\nu = k_\mu k_\nu \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} = k^2$$
$$\not{k}\not{\mathcal{D}}_x + \not{\mathcal{D}}_x\not{k} = (k^\mu \mathcal{D}_x^\nu + \mathcal{D}_x^\mu k^\nu)\frac{1}{2}\{\gamma_\mu,\gamma_\nu\} = 2k_\mu \mathcal{D}_x^\mu$$

(note that k and *D* commute).

We need to expand $$\mathcal{F}\left(k^2 - 2i\frac{k\cdot\mathcal{D}_x}{\mathcal{M}} - \left(\frac{\not{\mathcal{D}}_x}{\mathcal{M}}\right)^2\right)$$ in powers of $1/M$

We expect all powers lower than 4 to give zero , power 4 gives result independent of M higher powers vanish in the limit $M\to+\infty$ Moreover we need 4 gamma matrices to get non-zero result from the Dirac trace. This means that only second term in Taylor expansion is needed.

Chiral anomaly in gauge theory

$$\mathcal{A}(x) = -2 \lim_{M \to +\infty} M^4 \int \frac{d^4k}{(2\pi)^4} \operatorname{tr} \left\{ \gamma^5 \, t \, \mathcal{F}\left(-\left[i\not{k} + \frac{\not{D}_x}{M} \right]^2 \right) \right\}$$

Expanding:

$$\mathcal{F}\left(k^2 - 2i \frac{k \cdot \mathcal{D}_x}{\mathcal{M}} - \left(\frac{\not{\mathcal{D}}_x}{\mathcal{M}} \right)^2 \right) \to \frac{1}{2} \mathcal{F}''\left(k^2\right) \frac{\not{\mathcal{D}}_x^4}{\mathcal{M}^4} \qquad \mathcal{F}'(k^2) = \frac{d}{dk^2}\mathcal{F}(k^2)$$

we get

$$\mathcal{A}(x) = -\int \frac{d^4k}{(2\pi)^4} \mathcal{F}''(k^2) \operatorname{tr}\left(\gamma^5 \, t \not{D}_x^4 \right)$$

We can now integrate over d^4k

Chiral anomaly in gauge theory

$$\mathcal{A}(x) = -2 \lim_{M\to+\infty} M^4 \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left\{\gamma^5\, t\, \mathcal{F}\left(-\left[i\not{k} + \frac{\not{D}_x}{M}\right]^2\right)\right\}$$

Expanding:

$$\mathcal{F}\left(k^2 - 2i\frac{k\cdot\mathcal{D}_x}{\mathcal{M}} - \left(\frac{\not{\mathcal{D}}_x}{\mathcal{M}}\right)^2\right) \to \frac{1}{2}\mathcal{F}''\left(k^2\right)\frac{\not{\mathcal{D}}_x^4}{\mathcal{M}^4} \qquad \mathcal{F}'(k^2) = \frac{d}{dk^2}\mathcal{F}(k^2)$$

we get

$$\mathcal{A}(x) = -\int \frac{d^4k}{(2\pi)^4}\, \mathcal{F}''(k^2) \operatorname{tr}\left(\gamma^5\, t\not{D}_x^4\right)$$

We can now integrate over d^4k

$$\begin{aligned}
&\mathcal{F}(0) = 1\,,\\
&\mathcal{F}(+\infty) = 0\,,\\
&s\,\mathcal{F}'(s) = 0 \;\text{ at } s = 0 \;\text{ and at } s = +\infty
\end{aligned}$$

Chiral anomaly in gauge theory

Integration over d^4k

Go to Euclidean metric (lecture 2)

$$k^0 = ik_4 \quad \rightarrow \quad d^4k = dk^0 dk^1 dk^2 dk^3 = i dk^0 dk^1 dk^2 dk^3 = i d^4 k_E$$

$$\rightarrow \quad k^2 = \left(k^0\right)^2 - \sum_{i=1}^{3} (k^i)^2 = -\left(k_4\right)^2 - \sum_{i=1}^{3} (k^i)^2 = -k_E^2$$

$$\int d^4k\, \mathcal{F}''(k^2) \underset{k^0 = ik_4}{=} i \int d^4k_E\, \mathcal{F}''(-k_E^2)$$

$$= i\, 2\pi^2 \int_0^\infty dk\, k^3 \mathcal{F}''(-k^2)$$

$$\underset{x=-k^2}{=} -i\pi^2 \int_0^\infty dx\, x \mathcal{F}''(x)$$

$$= i\pi^2 \int_0^\infty dx\, \mathcal{F}'(x) = -i\pi^2$$

Chiral anomaly in gauge theory

Squaring covariant derivative:

$$
\begin{aligned}
\not{D}_x^2 &= D_x^\mu D_x^\nu \gamma_\mu \gamma_\nu \\
&= \frac{1}{2} D_x^\mu D_x^\nu \left(\{\gamma_\mu, \gamma_\nu\} + [\gamma_\mu, \gamma_\nu]\right) \\
&= D_x^2 + \frac{1}{4} [D_x^\mu, D_x^\nu] [\gamma_\mu, \gamma_\nu]
\end{aligned}
$$

Diggression

$$[D^\mu, D^\nu]\cdot\psi = (\partial^\mu - igA^\mu)(\partial^\nu - igA^\nu)\psi - (\partial^\nu - igA^\nu)(\partial^\mu - igA^\mu)\psi$$

$$=$$

$$= \underline{\partial^\mu\partial^\nu\psi} - ig\left(\partial^\mu A^\nu\right)\psi - ig\widehat{A^\nu\left(\partial^\mu\right.}\left.\psi\right) - \overline{igA^\mu\left(\partial^\nu\psi\right)} - g^2 A^\mu A^\nu\psi$$

$$- \underline{\partial^\nu\partial^\mu\psi} + ig\left(\partial^\nu A^\mu\right)\psi + \overline{igA^\mu\left(\partial^\nu\psi\right)} + ig\widehat{A^\nu\left(\partial^\mu\right.}\left.\psi\right) + g^2 A^\nu A^\mu\psi$$

$$= -ig\left\{\left(\partial^\mu A^\nu\right) - \left(\partial^\nu A^\mu\right)\right\}\psi - g^2\left[A^\mu, A^\nu\right]\psi \;=\; -igF^{\mu\nu}\psi$$

Chiral anomaly in gauge theory

$$\begin{aligned}
\not{D}_x^2 &= D_x^\mu D_x^\nu \gamma_\mu \gamma_\nu \\
&= \frac{1}{2} D_x^\mu D_x^\nu \left(\{\gamma_\mu, \gamma_\nu\} + [\gamma_\mu, \gamma_\nu]\right) \\
&= D_x^2 + \frac{1}{4} [D_x^\mu, D_x^\nu] [\gamma_\mu, \gamma_\nu] \\
&= D_x^2 - \frac{ig}{4} t^a F_a^{\mu\nu} [\gamma_\mu, \gamma_\nu]
\end{aligned}$$

We need a fourth power of $\not{D}_x$ traced with γ^5 so only a commutator squared survives.

Chiral anomaly in gauge theory

Calculating traces:

$$\begin{aligned}
\mathrm{tr}\left[\gamma^5 t\left(-\frac{ig}{4}t^a[\gamma_\mu,\gamma_\nu]F_a^{\mu\nu}\right)^2\right] &= -\frac{g^2}{16}\,\mathrm{Tr}\left(t\,t^a t^b\right)\mathrm{Tr}\left(\gamma^5\left[\gamma_\mu,\gamma_\nu\right]\left[\gamma_\rho,\gamma_\sigma\right]\right)F_a^{\mu\nu}F_b^{\rho\sigma} \\
&= -\frac{g^2}{4}\,\mathrm{Tr}\left(t\,t^a t^b\right)\underbrace{\mathrm{Tr}\left(\gamma^5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right)}_{-4i\varepsilon_{\mu\nu\rho\sigma}}F_a^{\mu\nu}F_b^{\rho\sigma} \\
&= ig^2\varepsilon_{\mu\nu\rho\sigma}F_a^{\mu\nu}F_b^{\rho\sigma}\,\mathrm{Tr}\left(t^a t^b t\right)
\end{aligned}$$

Puting things together

$$\begin{aligned}
\mathcal{A}(x) &= -\frac{1}{(2\pi)^4}\overbrace{\int d^4k\,\mathcal{F}''(k^2)}^{-i\pi^2}\quad\underbrace{\mathrm{tr}\left(\gamma^5 t\,\not{D}_x^4\right)}_{ig^2\varepsilon_{\mu\nu\rho\sigma}F_a^{\mu\nu}F_b^{\rho\sigma}\,\mathrm{Tr}\left(t^a t^b t\right)} \\
&= -\frac{g^2}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_a^{\mu\nu}(x)F_b^{\rho\sigma}(x)\,\mathrm{Tr}\left(t^a t^b t\right)
\end{aligned}$$

Chiral anomaly in gauge theory

Calculating traces:

$$\mathrm{tr}\left[\gamma^5 t\left(-\frac{ig}{4}t^a[\gamma_\mu,\gamma_\nu]F_a^{\mu\nu}\right)^2\right] = -\frac{g^2}{16}\mathrm{Tr}\left(t\,t^a t^b\right)\mathrm{Tr}\left(\gamma^5[\gamma_\mu,\gamma_\nu]\,[\gamma_\rho,\gamma_\sigma]\right)F_a^{\mu\nu}F_b^{\rho\sigma}$$

$$= -\frac{g^2}{4}\mathrm{Tr}\left(t\,t^a t^b\right)\underbrace{\mathrm{Tr}\left(\gamma^5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\right)}_{-4i\varepsilon_{\mu\nu\rho\sigma}}F_a^{\mu\nu}F_b^{\rho\sigma}$$

$$= ig^2\varepsilon_{\mu\nu\rho\sigma}F_a^{\mu\nu}F_b^{\rho\sigma}\,\mathrm{Tr}\left(t^a t^b t\right)$$

Puting things together

- in QED no trace
- if $t = 1$ the integral of $\mathcal{A}(x)$ is an integer Chern-Pontryagin index that charaterizes topological properties if the gluon field

$$\mathcal{A}(x) = -\frac{1}{(2\pi)^4}\overbrace{\int d^4k\,\mathcal{F}''(k^2)}^{-i\pi^2}\quad\underbrace{\mathrm{tr}\left(\gamma^5 t\,\not{D}_x^4\right)}_{ig^2\varepsilon_{\mu\nu\rho\sigma}F_a^{\mu\nu}F_b^{\rho\sigma}\,\mathrm{Tr}\left(t^a t^b t\right)}$$

$$= -\frac{g^2}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_a^{\mu\nu}(x)F_b^{\rho\sigma}(x)\,\mathrm{Tr}\left(t^a t^b t\right)$$

Anomaly of the axial current

Remember that the free lagrangian changes due to the anomaly in the following way

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha(x)\mathcal{A}(x)$$

If we add a source we get an extra term

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha(x)\mathcal{A}(x) + J_5^\mu(x)\,\partial_\mu\alpha(x)$$

We need to integrate this to get the action, last term integrate by parts and require that the total change of action is zero:

$$\langle \partial_\mu J_5^\mu(x) \rangle_A = -\frac{g^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}\,F_a^{\mu\nu}(x)\,F_b^{\rho\sigma}(x)\,\mathrm{tr}\,(t^a t^b t)$$

where $\langle\cdot\rangle_A$ is an average over the fermion fields, in a fixed gauge field configuration.

Anomaly in the light quark sector

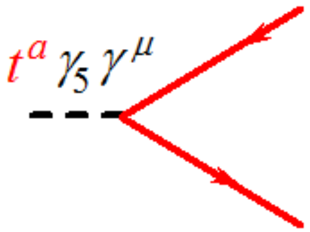
Recall Noether theorem:
global symmetry implies conserved current(s)
To calculate conserved currents promote the symmetry to the local one,
calculate the change of action (as disscused on previous slide).

Consider SU(2) chiral transformation:

$$\mathcal{U}(x) = \exp\left(i\gamma^5 \alpha^a(x)\, t^a\right) \qquad \psi = \begin{bmatrix} u \\ d \end{bmatrix}$$

Conserved current:

$$J_5^{\mu\, a} = \bar{\psi}\gamma^5\gamma^\mu t^a \psi$$



Anomaly in the light quark sector

Consider diagonal (neutral) axial current generated by matrix

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in falovor space and unit matrix in the color (gauge) space. Then:

$$\mathrm{tr}\left(t^a t^b t\right) = \mathrm{tr}_{\mathrm{colour}}\left(t^a t^b\right) \times \underbrace{\mathrm{tr}_{\mathrm{flavour}}\left(t\right)}_{1-1=0} = 0$$

Anomaly vanishes. Physically up quark contribution is cancelled by d quark.

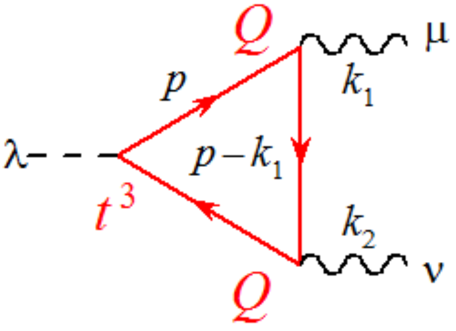
Anomaly in the light quark sector

Consider diagonal (neutral) axial current generated by matrix

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

coupled to QED. In flavor space $\psi = \begin{bmatrix} u \\ d \end{bmatrix}$ electric charge is a matrix

$$Q \equiv \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$$



therefore anomaly is proportional to

$$\mathrm{tr}_{\mathrm{flavour}}\left(Q^2 t\right) \times \mathrm{tr}_{\mathrm{colour}}\left(\mathbf{1}_{\mathrm{colour}}\right) = \frac{N_c}{3}$$

Atiyah-Singer theorem

Dirac matrices: $$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}$$

hermitean antihermitean

Dirac operator is neither hermitean not antihermitean. Let's go to Euclidean space

$$x^0 \;=\; ix^4 \to \partial_0 = \frac{\partial}{\partial x^0} = -i\frac{\partial}{\partial x^4} = -i\partial_4 \qquad A^0 \;=\; iA^4 \qquad \gamma^0 \;=\; i\gamma^4$$

Then:
$$\begin{aligned} \not{D}_x &= \gamma^0\partial_0 + \gamma^k\partial_k - ig\left(A_a^0\gamma^0 - A_a^k\gamma^k\right)t^a \\ &= \gamma^4\partial_4 + \gamma^k\partial_k + ig\left(A_a^4\gamma^4 + A_a^k\gamma^k\right)t^a \\ &= \sum_{j=1}^{4}\left(\partial_j + igA_a^j t^a\right)\gamma^j \end{aligned}$$

is hermitean becuse all gamma matrices are antihermitean.

Atiyah-Singer theorem

Dirac oprtator in Euclidean space can be therefore diagonalized in an orthonormal basis of eigenfunctions ϕ_k

$$\not{D}_x \phi_k(x) = \lambda_k \phi_k(x) \,,$$

$$\int d^4x_E \; \phi_k^\dagger(x)\phi_{k'}(x) = \delta_{kk'}$$

$$\sum_k \phi_k(x)\phi_k^\dagger(y) = \delta(x-y)$$

Anomaly function

$$\mathcal{A}(x) = -2 \lim_{y\to x, M\to+\infty} \operatorname{tr}\left\{\gamma^5 \, t \, \mathcal{F}\left(-\frac{\not{D}_x^2}{M^2}\right)\right\}\delta(x-y)$$

for $t = 1$

can be rewritten as:

$$\begin{aligned}
\mathcal{A}(x) &= -2 \lim_{y\to x, M\to+\infty} \operatorname{tr}\left\{\gamma^5 \mathcal{F}\left(-\frac{\not{D}_x^2}{M^2}\right)\sum_k \phi_k(x)\phi_k^\dagger(y)\right\} \\
&= -2 \lim_{y\to x, M\to+\infty} \sum_k \operatorname{tr}\left\{\phi_k^\dagger(y)\,\gamma^5 \mathcal{F}\left(-\frac{\not{D}_x^2}{M^2}\right)\phi_k(x)\right\} \\
&= -2 \lim_{M\to+\infty} \sum_k \mathcal{F}\left(-\frac{\lambda_k^2}{M^2}\right)\phi_k^\dagger(x)\gamma^5\phi_k(x) \,.
\end{aligned}$$

Atiyah-Singer theorem

We can connect this result with the previous one, rewritten in Euclidean metric

$$\frac{g^2}{32\pi^2}\int d^4x_E\ \epsilon_{ijkl}\,F^a_{ij}(x)\,F^b_{kl}(x)\,\mathrm{tr}(t^a t^b)$$
$$= -\frac{1}{2}\int d^4x_E\ \mathcal{A}(x) = \lim_{M\to+\infty}\sum_k \mathcal{F}\left(-\frac{\lambda_k^2}{M^2}\right)\int d^4x_E\ \phi_k^\dagger(x)\gamma^5\phi_k(x)$$

We can relate eigenvalues of $\phi_k(x)$ to eigenvalues of $\gamma^5\phi_k(x)$
since $\{\gamma^5, \not{D}\} = 0$

So have $\not{D}_x(\gamma^5\phi_k(x)) = -\lambda_k(\gamma^5\phi_k(x))$ This means that for $\lambda_k \neq 0$
functions $\phi_{k'} \equiv \gamma^5\phi_k$ and ϕ_k are different eigenfunctions of $\not{D}_x$ hence

$$\int d^4x_E\ \phi_k^\dagger(x)\gamma^5\phi_k(x) = \int d^4x_E\,\phi_k^\dagger(x)\phi_{k'}(x) = 0$$

Therefore only eigenfunctions with $\lambda_k = 0$ so called zero modes contribute
to the anomaly.

Atiyah-Singer theorem

Anomaly expressed in terms of the zero modes

$$\frac{g^2}{32\pi^2}\int d^4x_E\ \epsilon_{ijkl}\,F^a_{ij}(x)\,F^b_{kl}(x)\,\mathrm{tr}(t^a t^b) = \sum_{k|\lambda_k=0}\int d^4x_E\ \phi^\dagger_k(x)\gamma^5\phi_k(x)$$

Since $\{\gamma^5, \not{D}_x\} = 0$ zero modes can be chosen to be also eigenstates of γ^5
so called left and right zero modes

$$\not{D}_x\phi_R(x) = 0\,, \quad \gamma^5\phi_R(x) = +\phi_R(x)$$
$$\not{D}_x\phi_L(x) = 0\,, \quad \gamma^5\phi_L(x) = -\phi_L(x)$$

Because zero modes are normalized

$$\frac{g^2}{32\pi^2}\int d^4x_E\ \epsilon_{ijkl}\,F^a_{ij}(x)\,F^b_{kl}(x)\,\mathrm{tr}(t^a t^b) = n_R - n_L$$

where n_R and n_L are numbers of right and left zero modes. The difference is an integer. This formula is called Attiyah-Singer index theorem.
There exist nonperturbative, nontrivial configurations of the gauge field with above property – instantons.