QCD lecture 4

November 4

Infrared divergences

$$S_F^R = \frac{i}{\not p}\left(1 + \frac{\alpha(\mu^2)}{4\pi} C_F \left(\ln\left(\frac{-p^2}{\bar{\mu}^2}\right) - 1\right)\right)$$

Divergent for $p^2 = 0$. This is infrared divergence (from the lower int. limit).
It can be regularized by going to the number of dimensions higher than 4.
Before expansion, change $\varepsilon \to -\kappa$

$$S_F^R(p) \;\; = \;\; \frac{i}{\not p}\left(1 - \frac{\alpha_s}{4\pi} C_F \left(\frac{\bar{\mu}^2}{-p^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon} + 1\right) + \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon}\right)$$

Infrared divergences

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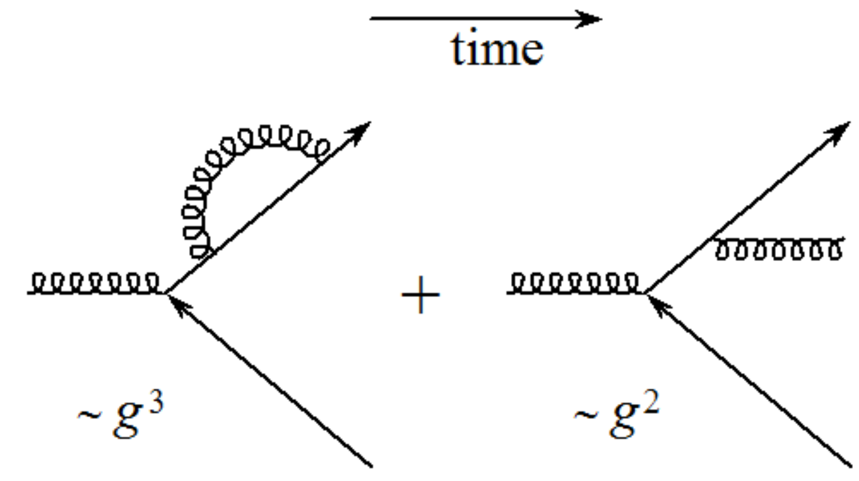
Before expansion, change $\varepsilon \to -\kappa$

$$\begin{aligned} S_F^R(p) &= \frac{i}{\not{p}}\left(1 - \frac{\alpha_s}{4\pi} C_F \left(\frac{-p^2}{\bar{\mu}^2}\right)^{\kappa}\left(-\frac{1}{\kappa} + 1\right) - \frac{\alpha_s}{4\pi} C_F \frac{1}{\kappa}\right) \\ &\underset{p^2=0}{=} \frac{i}{\not{p}}\left(1 - \frac{\alpha_s}{4\pi} C_F \frac{1}{\kappa}\right). \end{aligned}$$

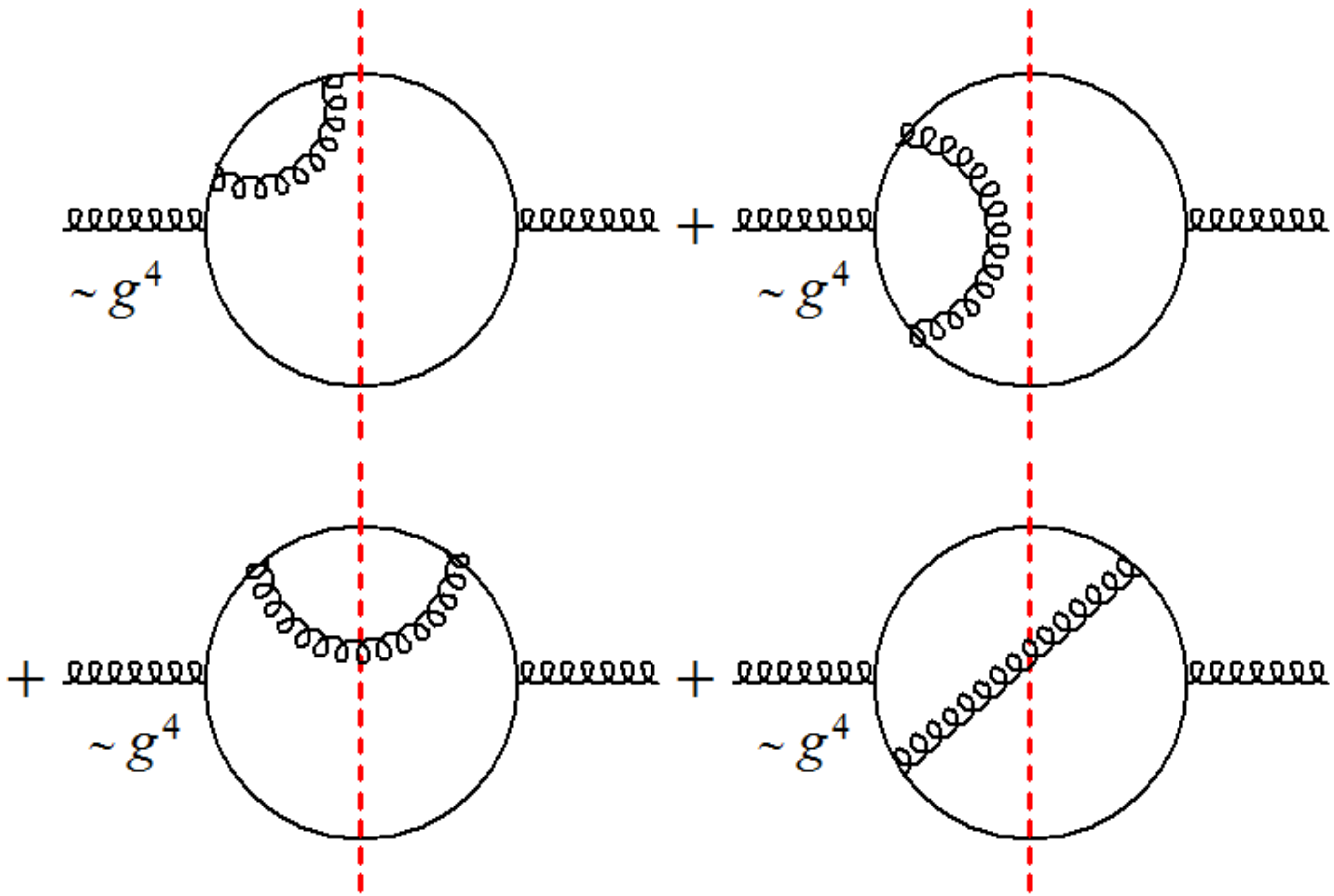
Infrared divergencies



One cannot distinguish a single electron from an electron accompanied by a zero energy foton or a collinear foton (for massless fermion).

One has to sum over such degenerate states.

Infrared divergencies



Here IR singularities cancel out

Infrared singularities

IR singulariteis arise when the theory has massless particles (photon, gluon)

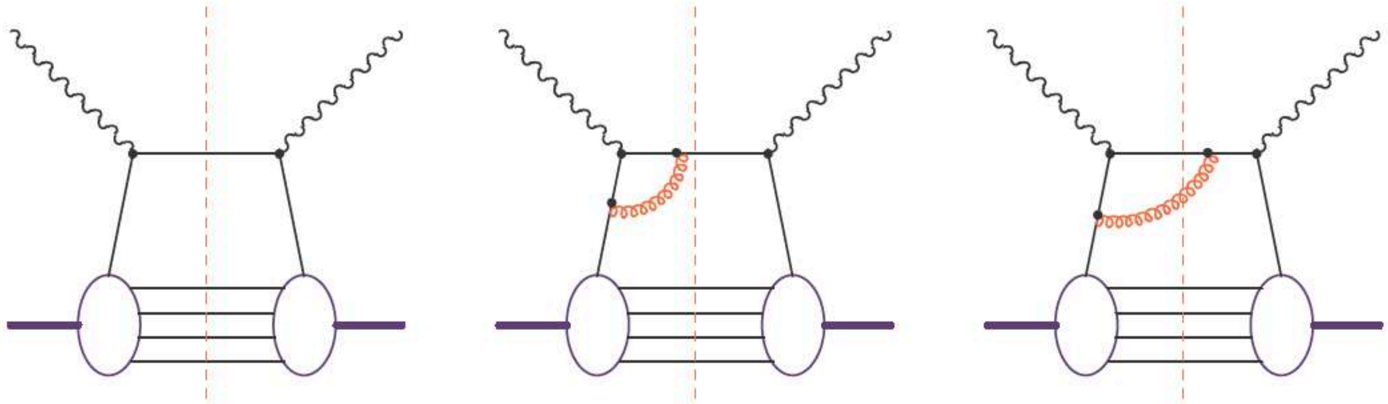
- when energy of photon (gluon) is small – soft singularity
- when for massless fermion photon (gluon) is parallel to that fermion – collinear singularity

Bloch – Nordsieck theorem (baically derived for QED)
Kinoshita – Lee – Nauenberg theorem (generalized to QCD)

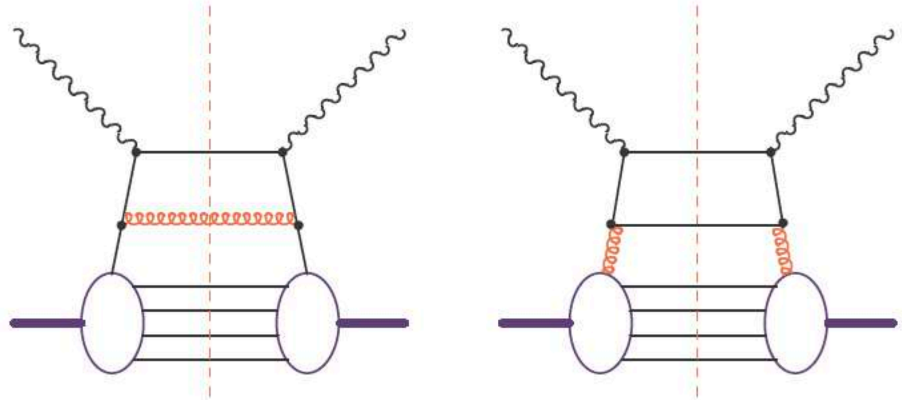
Kinoshita-Lee-Nauenberg (KLN) theorem assures that a summation over degenerate initial and final states removes all infrared (IR) divergences in QCD.

This very broad topic, beyond the scope of this lecture

QCD corrections to parton model



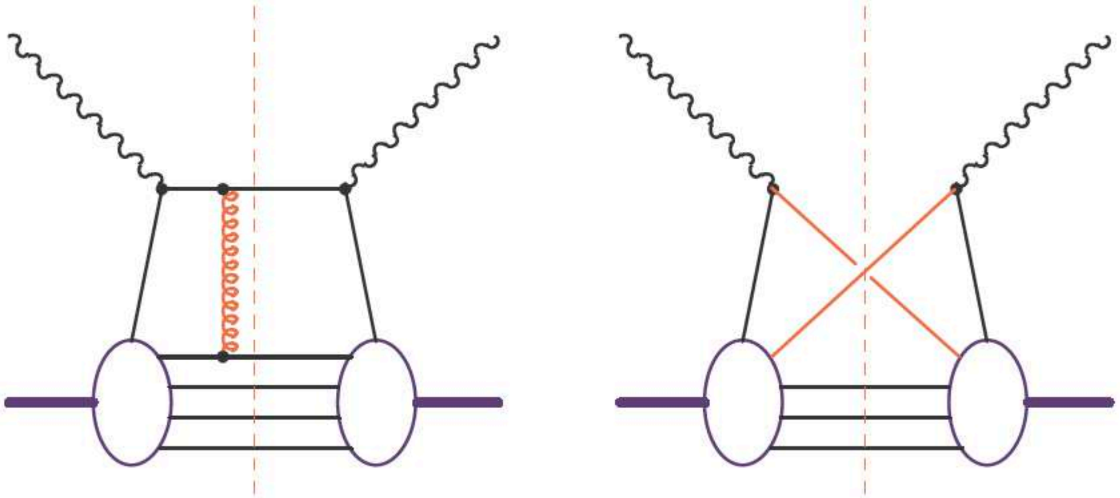
Leading corrections
not suppressed by $1/Q^2$



photon scatters off the gluon

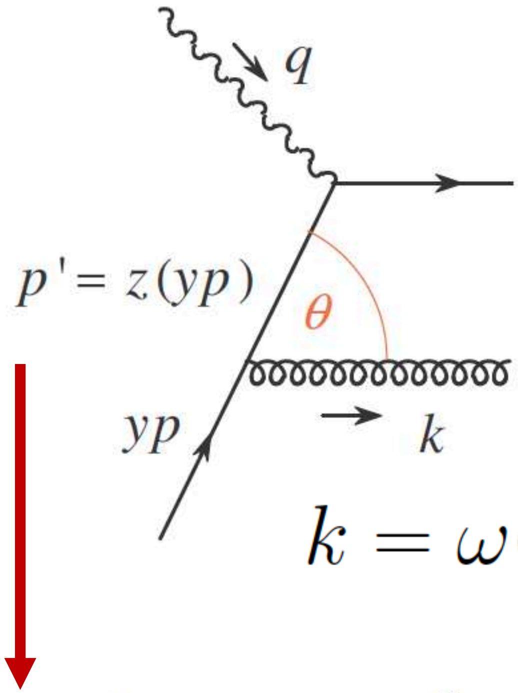
QCD corrections to parton model

Non-leading corrections suppressed by $1/Q^2$



QCD corrections to parton model

$$yp = E(1,0,0,1)$$



$$d^4k \longrightarrow d\omega d\cos\theta$$

$$k = \omega(1,\ \sin\theta\sin\varphi,\ \sin\theta\cos\varphi,\ \cos\theta)$$

$$\frac{1}{p'^{\,2}} = \frac{1}{(yp-k)^2} = \frac{1}{2yp\,k} = \frac{1}{2E\omega(1-\cos\theta)}$$

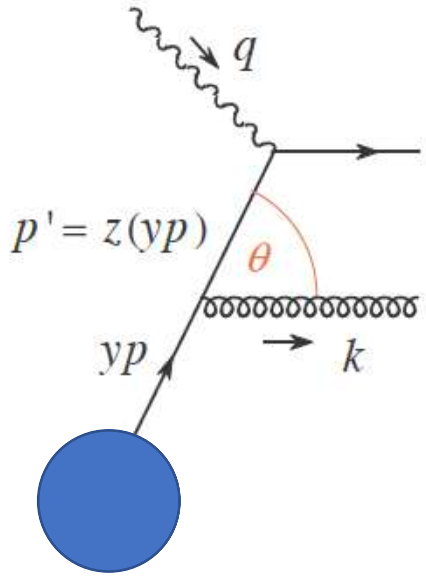
QCD corrections to parton model

$$\frac{d\omega d\cos\theta}{2E\omega(1-\cos\theta)}$$

- soft (cancel) $\omega \longrightarrow 0$
- collinear (remain) $\theta \longrightarrow 0$

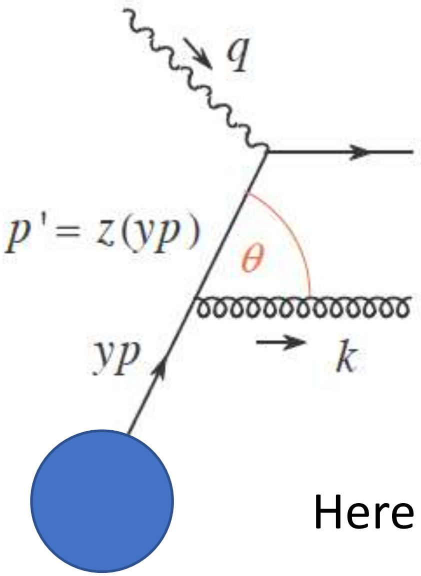
In dimensional regularization:

$$\left(\frac{Q^2}{\mu^2}\right)^{\kappa}\frac{1}{\kappa} = \frac{1}{\kappa} + \log\left(\frac{Q^2}{\mu^2}\right)$$



Poles can be absorbed into bare parton densities.
Logs can be summed up to all orders. Factrozation.
Coefficients of the poles are universal functions of z

Altarelli-Parisi probabilities



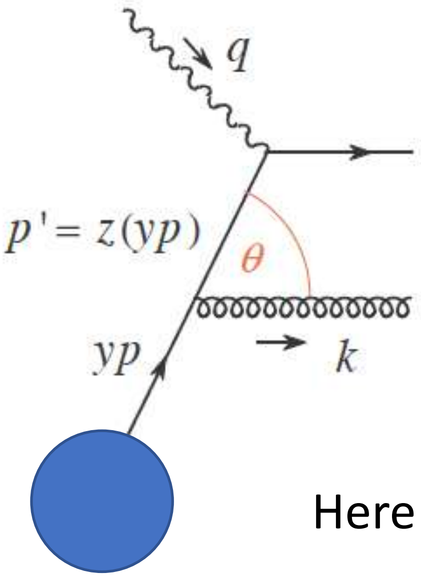
$$\log\left(\frac{Q^2}{\mu^2}\right) P_{qq}(z)$$

It turns out that potentially large logs are multiplied by universal functions of the momentum fraction *z* (with respect to the emitting parton)

Here $P_{qq}(z) = P_{q\leftarrow q}(z)$ is a probability of "finding"

a quark of the longitudinal momentum fraction z in initial quark

Altarelli-Parisi probabilities



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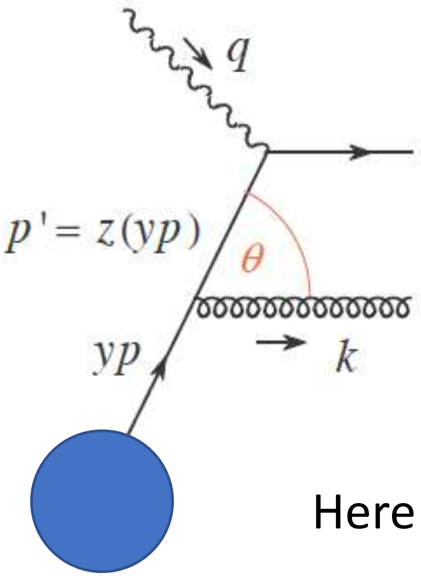
$$P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+$$

Altarelli-Parisi probabilities



sample diagram

$$\log\left(\frac{Q^2}{\mu^2}\right) P_{qq}(z)$$

It turns out that potentially large logs are multiplied by universal functions of the momentum fraction *z* (with respect to the emitting parton)

Here $P_{qq}(z) = P_{q\leftarrow q}(z)$ is a probability of "finding"

a quark of the longitudinal momentum fraction z in initial quark

$$P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)_+$$

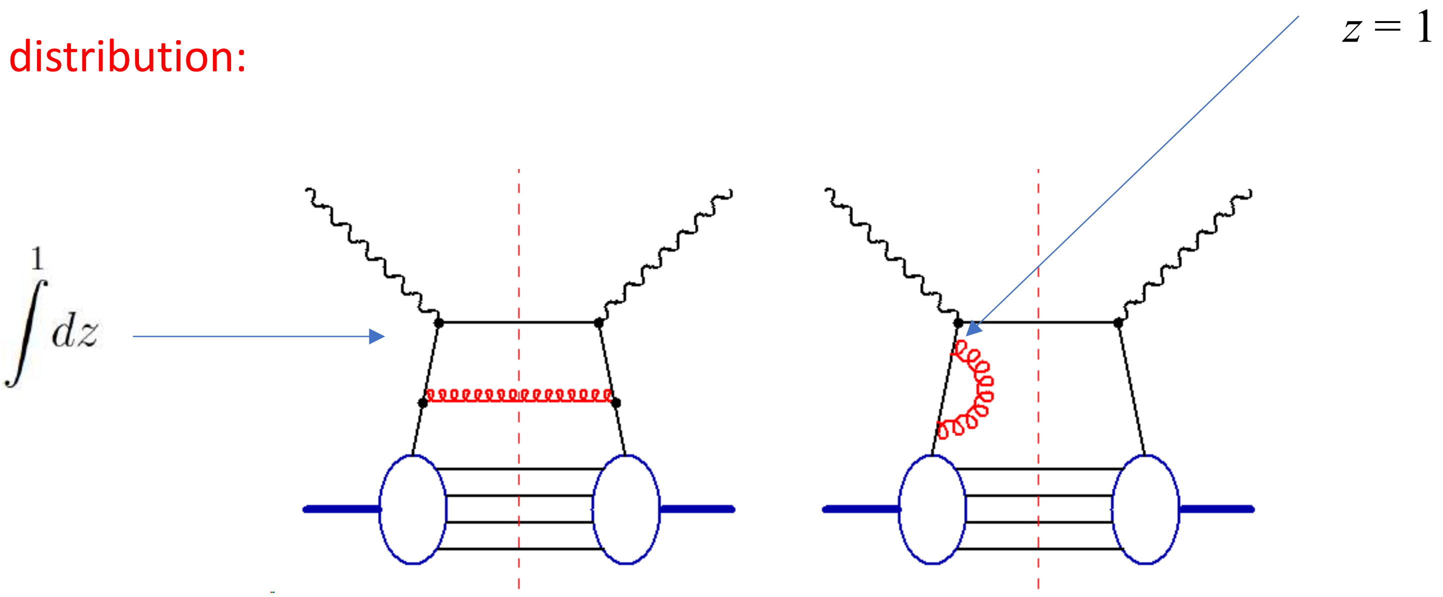
"Plus" distribution:

$$\int^1 dz\,(\ldots)_+\,g(z) = \int^1 dz\,(\ldots)\,[g(z)-g(1)]$$

appears because of the virtual diagram for which $z = 1$

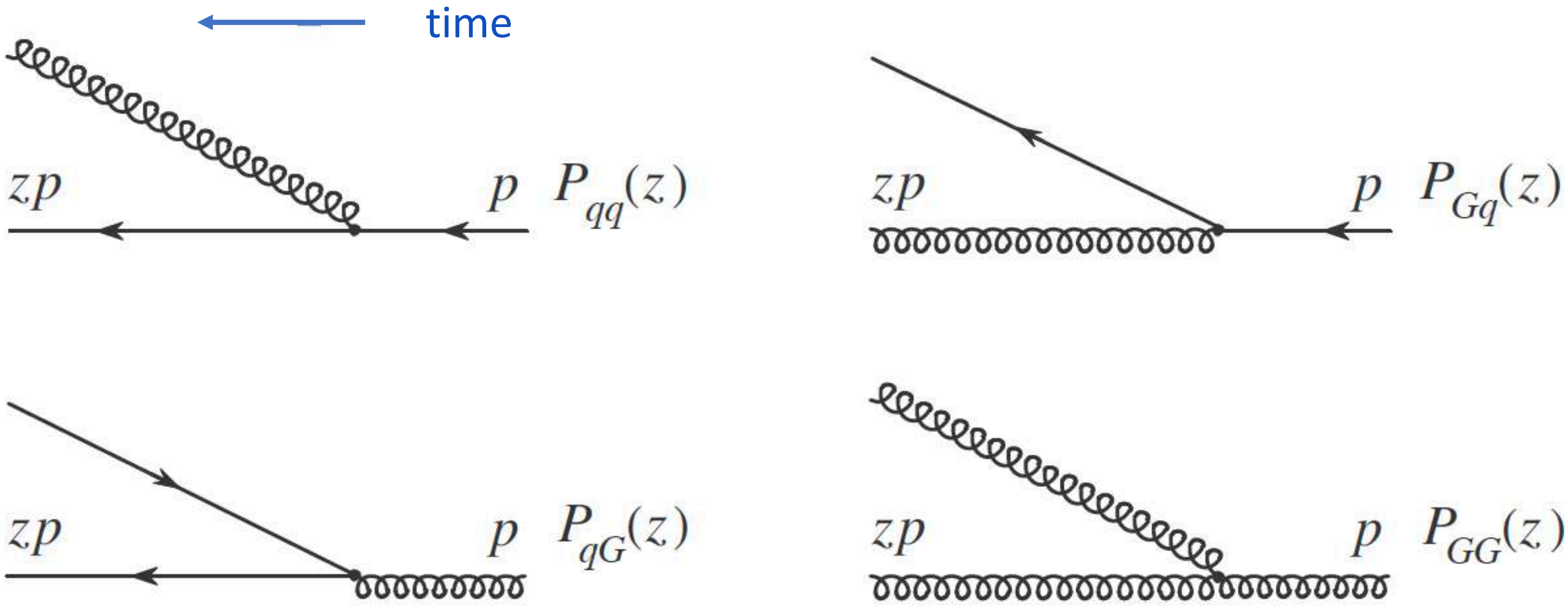
Altarelli-Parisi probabilities

"Plus" distribution:



Different diagrams give extra contribution at $z = 1$ in different gauges.
The result is the same: no singularity at $z = 1$.

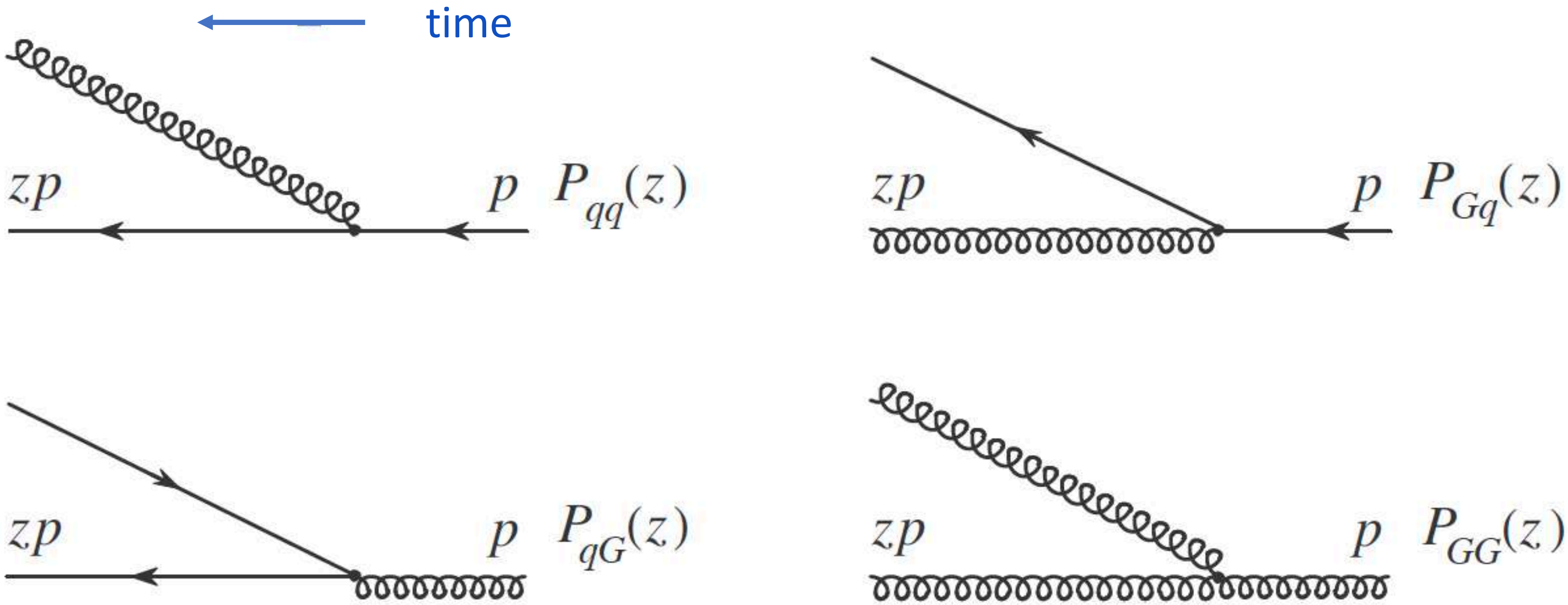
Altarelli-Parisi probabilities



$$P_{qq}(z) = C_F \left(\frac{1+z^2}{1-z}\right)_+, \quad P_{Gq}(z) = C_F \frac{1+(1-z)^2}{z}, \quad P_{qG}(z) = \frac{1}{2}\left[z^2 + (1-z)^2\right]$$

$$P_{GG}(z) = 2C_A\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + \frac{1}{2}\left(\frac{11}{3}C_A - \frac{2}{3}n_f\right)\delta(1-z)$$

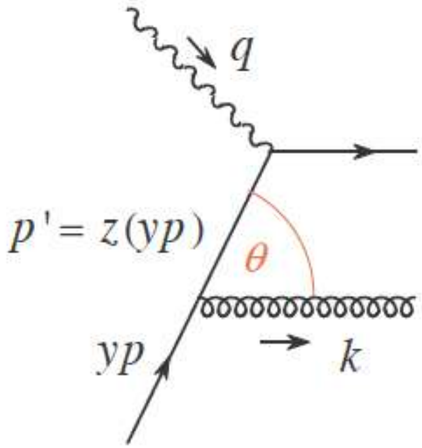
Altarelli-Parisi probabilities



$$P_{qG}(z) = P_{\overline{q}G}(z), \qquad P_{Gq}(z) = P_{G\overline{q}}(z),$$

$$P_{qq}(z) = P_{Gq}(1-z), \; P_{GG}(z) = P_{GG}(1-z), \quad P_{qG}(z) = P_{qG}(1-z$$

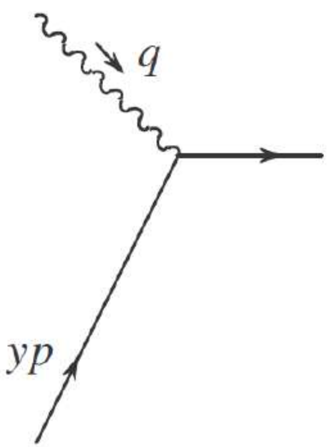
QCD corrections to parton model



on-shell condition

$$0 = (zyp + q)^2 = 2zy\, pq + q^2 = 2M\nu\, zy - Q^2$$

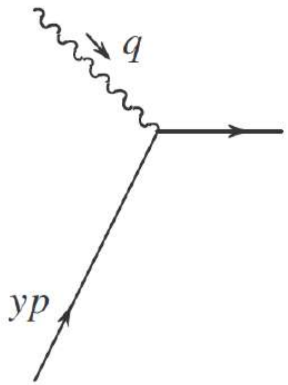
$$zy = \frac{Q^2}{2M\nu} = x$$



Recall F_1: $$F_1(x) = \frac{1}{2}\sum_i e_i^2 f_i(x)$$

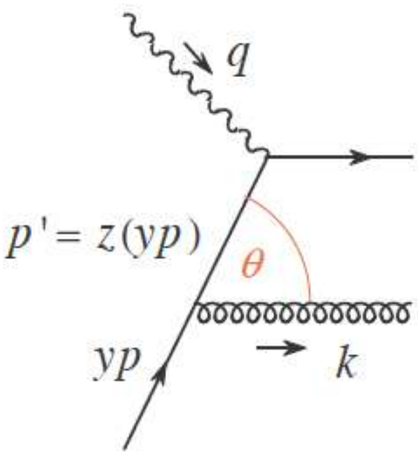
$$2F_1(x) = e_q^2 \int_0^1 dyq(y)\delta(y - x)$$

QCD corrections to parton model



Recall F_1:

$$2F_1(x) = e_q^2 \int_0^1 dy q(y) \delta(y - x)$$

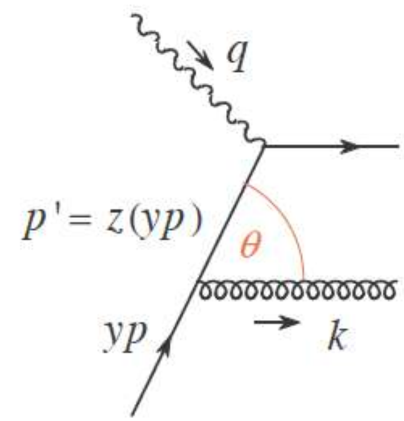


$$2\Delta F_1(x) = e_q^2 \frac{\alpha_s}{2\pi} \int_0^1 dy q(y) \int_0^1 dz \delta(zy - x) \left[P_{qq}(z) \ln \frac{Q^2}{\mu^2} + C(z) \right]$$

Correction to F_1 large logs

$$q(x,Q^2) = q(x,\mu^2) + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu^2}\int_x^1 \frac{dy}{y} P_{qq}\left(\frac{x}{y}\right) q(y,\mu^2) + \ldots$$

$$= q(x,\mu^2) + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu^2} P_{qq} \otimes q(\mu^2) \text{ -}$$



Convolution:

$$P_{qq} \otimes q = \int_0^1 dz \int_0^1 dy\,\delta(zy - x) P_{qq}(z) q(y)$$

Integration over $d\theta$ gave a pole

DGLAP Evolution Equation

$$\frac{d}{d\ln Q^2} = Q^2 \frac{d}{dQ^2} \Rightarrow q(x, Q^2) = q(x, \mu^2) + \frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2} P_{qq} \otimes q(\mu^2) + \ldots$$

Evolution eq.
Dokshitzer,
Gribov, Lipatov
Altarelli, Parisi

$$\frac{d}{d\ln Q^2} q(x, Q^2) = \frac{\alpha_s}{2\pi} P_{qq} \otimes q(Q^2)$$

Such equation sums up all powers $\frac{\alpha_s}{2\pi} \ln \frac{Q^2}{\mu^2}$.

Leading Log Approximation (LLA)

DGLAP Evolution Equations

Full set of DGLAP equations:

$$Q^2 \frac{d}{dQ^2} q_i(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{qq} \otimes q_i(Q^2) + P_{qG} \otimes G(Q^2)\right]$$

$$Q^2 \frac{d}{dQ^2} G(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{Gq} \otimes \sum_i q_i(Q^2) + P_{GG} \otimes G(Q^2)\right]$$

We need an input at one scale Q_0^2 and then we can evolve them up to some other Q^2
note that index i runs over quarks and antiquarks
when we construct a difference, called non-singlet, gluons cancel

$$q_i^{NS}(x, Q^2) = q_i(x, Q^2) - \overline{q}_i(x, Q^2)$$

DGLAP Evolution Equations

Define:

singlet

$$q^S(x,Q^2) = \sum_i \left(q_i(x,Q^2) + \overline{q}_i(x,Q^2)\right)$$

nonsinglet

$$q_i^{NS}(x,Q^2) = q_i(x,Q^2) - \overline{q}_i(x,Q^2)$$
$$= q_i(x,Q^2) - \frac{1}{2n_f} q^S(x,Q^2)$$

DGLAP Evolution Equations

$$Q^2 \frac{d}{dQ^2} q^{NS}(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} P_{qq} \otimes q^{NS}(Q^2)$$

$$Q^2 \frac{d}{dQ^2} q^{S}(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{qq} \otimes q^{S}(Q^2) + 2n_f P_{qG} \otimes G(Q^2) \right]$$

$$Q^2 \frac{d}{dQ^2} G(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \left[P_{Gq} \otimes q^{S}(Q^2) + P_{GG} \otimes G(Q^2) \right]$$

DGLAP for Mellin moments

Moments of the convolution

$$M_n = \int_0^1 dx\, x^{n-1} P \otimes f = \int_0^1 dx\, x^{n-1} \int_0^1 dz \int_0^1 dy \delta(zy - x) P(z) f(y)$$

$$= \int_0^1 dz\, z^{n-1} P(z) \int_0^1 dy\, y^{n-1} f(y) = P_n\, f_n = \gamma^n f_n$$

γ^n anomalous dimension

convolution is replaced by a product

DGLAP for Mellin moments

$$\frac{dq_n^{NS}(t)}{dt} = \frac{\alpha_s(t)}{2\pi}\gamma_{qq}^n q_n^{NS}(t)$$

$$\frac{d}{dt}\begin{bmatrix} q_n^S(t) \\ G_n(t) \end{bmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{bmatrix} \gamma_{qq}^n & 2n_f\gamma_{qG}^n \\ \gamma_{Gq}^n & \gamma_{GG}^n \end{bmatrix}\begin{bmatrix} q_n^S(t) \\ G_n(t) \end{bmatrix}$$

$$\frac{\alpha_s(t)}{2\pi} = 2\,a_s(t) = 2\,\frac{1}{\beta_0 t}$$

Anomalous dimensions

$$\gamma_{qq}^{n} = C_F\left[-2\sum_{k=1}^{n+1}\frac{1}{k}+\frac{3}{2}+\frac{1}{n}+\frac{1}{n+1}\right],$$

$$\gamma_{qG}^{n} = \frac{1}{2}\frac{2+n+n^2}{n(n+1)(n+2)},$$

$$\gamma_{Gq}^{n} = C_F\frac{2+n+n^2}{n(n^2-1)}$$

$$\gamma_{Gq}^{n} = 2C_A\left[\frac{11}{12}-\sum_{k=1}^{n+2}\frac{1}{k}+\frac{1}{n-1}-\frac{1}{n}+\frac{1}{n+1}\right]-\frac{n_f}{3}$$

Valnce quark # conservation

$$\gamma_{qq}^{n} = C_F \left[-2 \sum_{k=1}^{n+1} \frac{1}{k} + \frac{3}{2} + \frac{1}{n} + \frac{1}{n+1} \right]$$

$$\gamma_{qq}^{1} = 0 \quad \rightarrow \quad \frac{dq_n^{NS}(t)}{dt} = 0$$

$$\int dx \left[q_i(x, Q^2) - \overline{q}_i(x, Q^2) \right] = \text{const.} = \int dx q_{Vi}(x, Q^2)$$

Momentum conservation

consider moment $n = 2$ for the singlet eqs.

$$\frac{d}{dt} q_2^S(t) = -\frac{2}{\beta_0 t}\left[\frac{4C_F}{3} q_2^S(t) - \frac{n_f}{3} G_2(t)\right] = -\frac{2}{\beta_0 t} f(t)$$

$$\frac{d}{dt} G_2(t) = +\frac{2}{\beta_0 t}\left[\frac{4C_F}{3} q_2^S(t) - \frac{n_f}{3} G_2(t)\right] = +\frac{2}{\beta_0 t} f(t)$$

$$q_2^S(t) + G_2(t) = \text{const.}$$

$$= \int dx\, x \left[\sum_i \left(q_i(x, Q^2) + \overline{q}_i(x, Q^2)\right) + G(x, Q^2)\right] = 1$$

value of 1 is a requirement for a proper normalization

Gluon momentum

$$\frac{d}{dt}q_2^S(t) = -\frac{2}{\beta_0 t}\left[\frac{4C_F}{3}q_2^S(t) - \frac{n_f}{3}G_2(t)\right] = -\frac{2}{\beta_0 t}f(t)$$

$$\frac{d}{dt}G_2(t) = +\frac{2}{\beta_0 t}\left[\frac{4C_F}{3}q_2^S(t) - \frac{n_f}{3}G_2(t)\right] = +\frac{2}{\beta_0 t}f(t)$$

Form a linear combination

$$\frac{4C_F}{3}\frac{d}{dt}q_2^S(t) - \frac{n_f}{3}\frac{d}{dt}G_2(t) = \frac{d}{dt}f(t) = -\frac{2}{\beta_0 t}\left[\frac{4C_F}{3} + \frac{n_f}{3}\right]f(t)$$

since $c = \frac{4C_F}{3} + \frac{n_f}{3} > 0$

the solution is trivial and tends to 0 $f(t) = f(t_0)\left(\frac{t}{t_0}\right)^{-2c/\beta_0} \underset{t\to\infty}{\longrightarrow} 0$

Gluon momentum

We have two asymptotic constraints:

$$f(t) = \frac{4C_F}{3} q_2^S(t) - \frac{n_f}{3} G_2(t) = 0 \qquad q_2^S(t) + G_2(t) = 1$$

which give

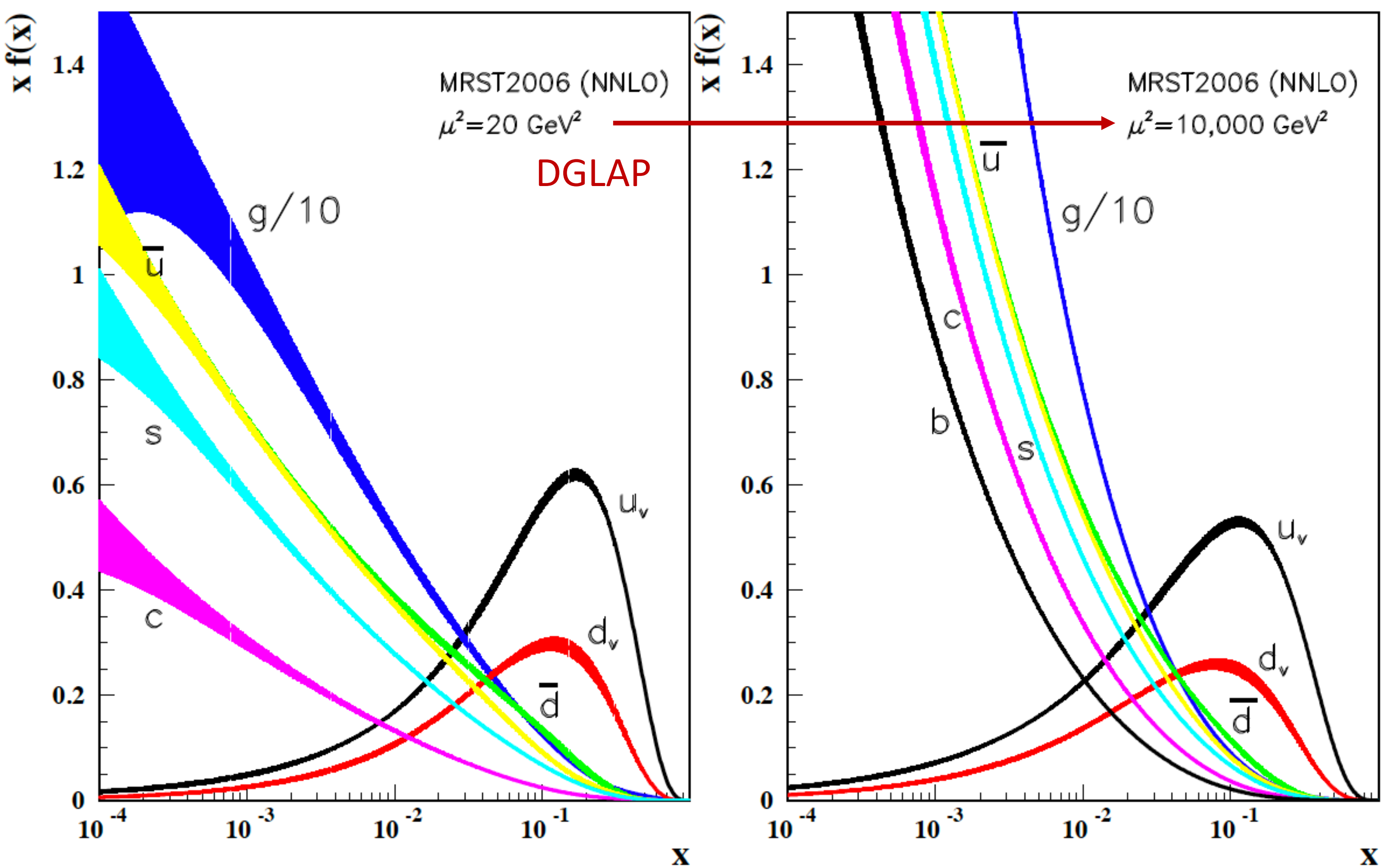
$$q_2^S(t) = \frac{n_f}{4C_F} G_2(t) \qquad \rightarrow \qquad \left[\frac{n_f}{4C_F} + 1\right] G_2(t) = 1$$

numerically we have

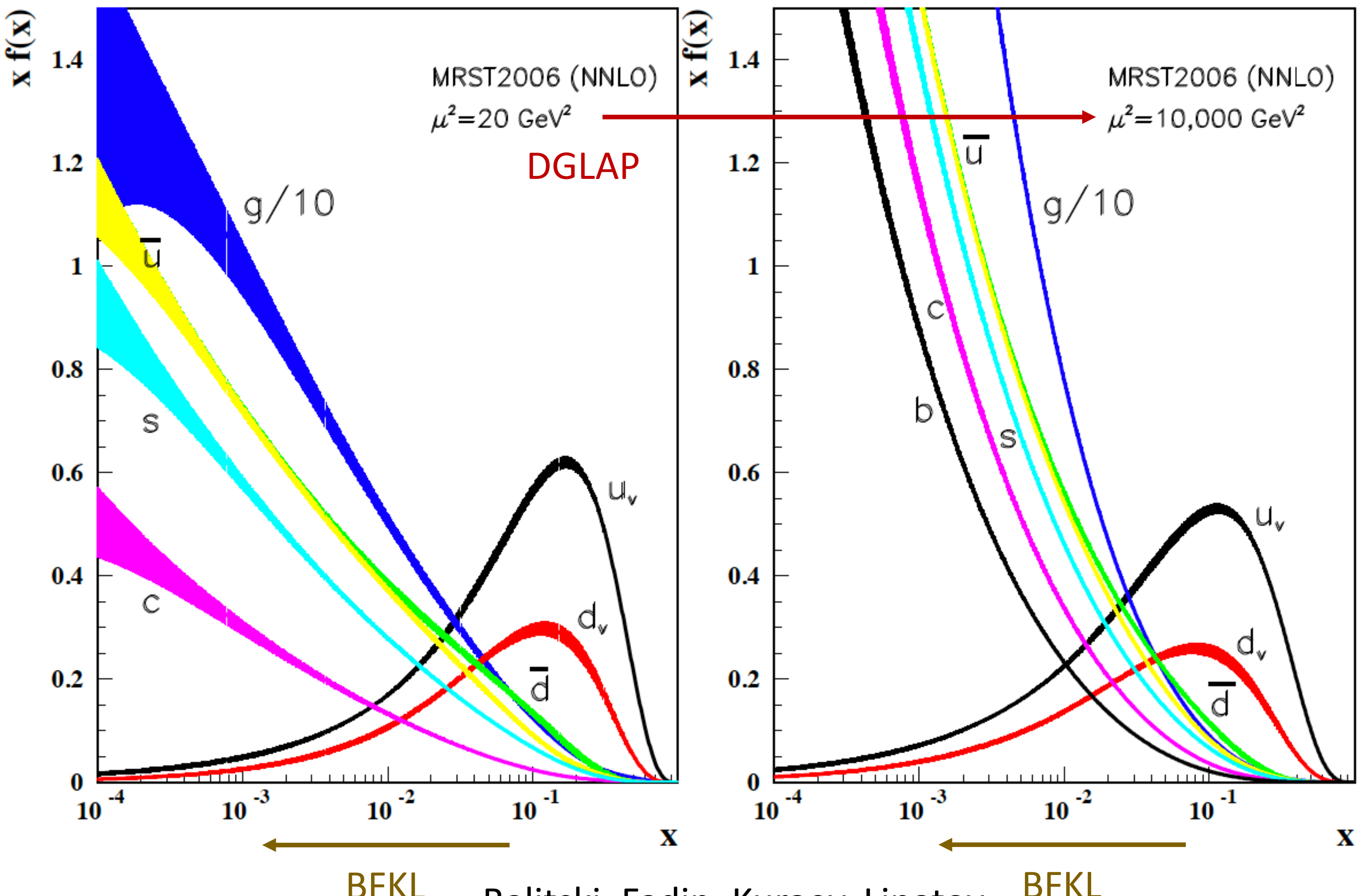
$$G_2(t) = \frac{1}{\frac{n_f}{4C_F} + 1} = \frac{16}{16 + 3n_f} = \underset{n_f=3}{0.64}, \underset{n_f=4}{0.57}, \underset{n_f=5}{0.52}, \underset{n_f=6}{0.47}$$

asymptotically gluons carry around 50% of total momentum!

Numerical solutions



Numerical solutions



Balitski, Fadin, Kuraev, Lipatov

HERA F_2: data vs. theory

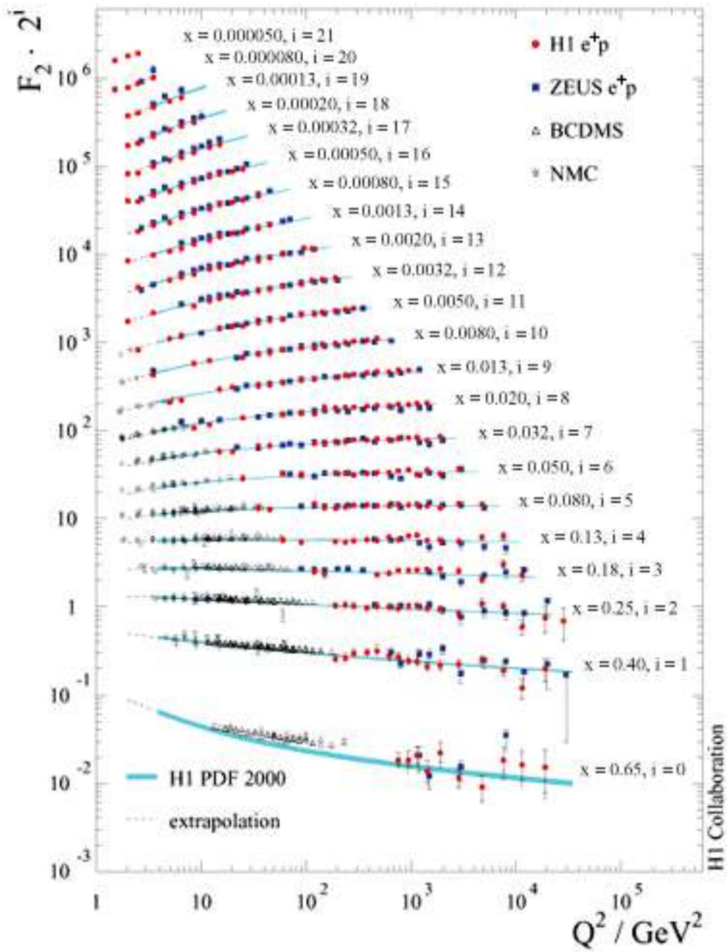


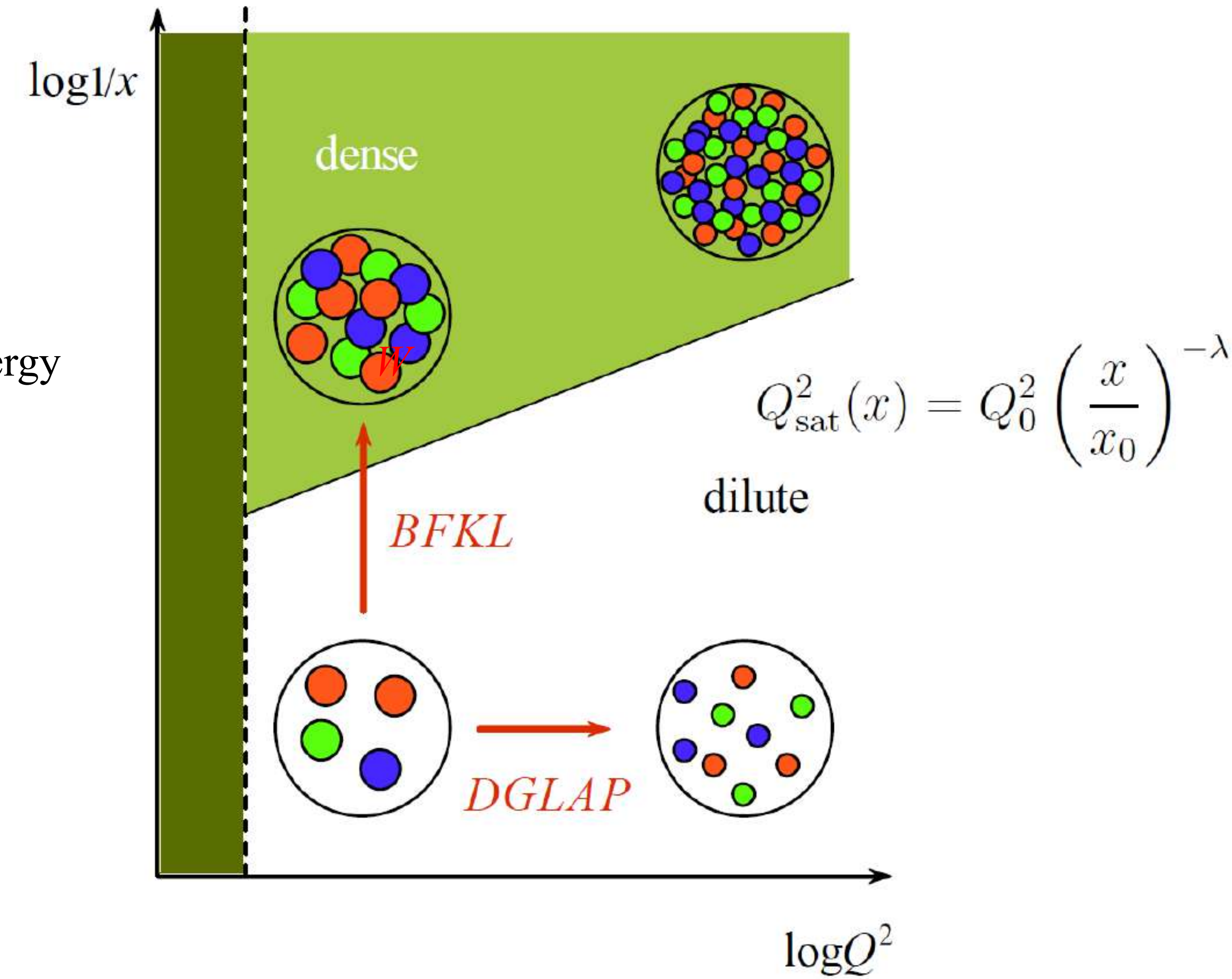
FIG. 2: Structure function F_2 as a function of Q^2 based on HERA-I measurements of H1 [2, 3] and ZEUS [4] collaboration compared to results from fixed target experiments BCDMS [5] and NMC [6].

DGLAP vs. BFKL

small x
large W

W – gamma+proton energy

large x
small W



Axial anomaly

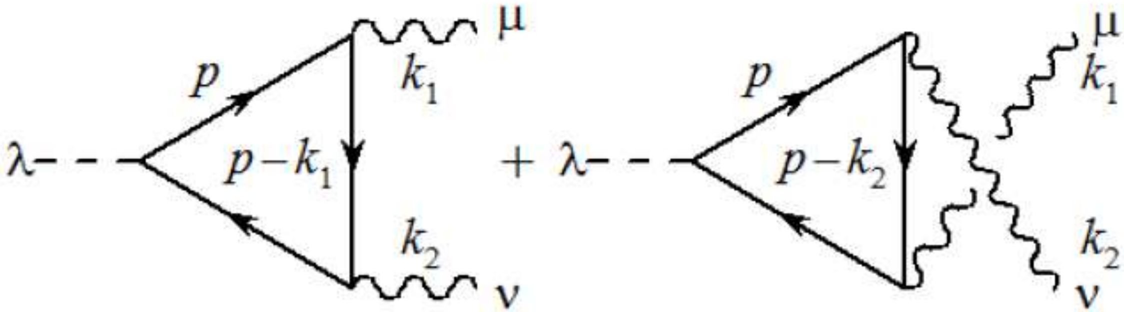
pseudoscalar density

Gauge invariance of QED (and QCD): $q_\mu j^\mu(q) = \bar{u}(p')\gamma^\mu u(p) = 0$

divergence of axial-vector current: $q_\mu j_5^\mu(q) = \bar{u}(p')\gamma^\mu\gamma_5 u(p) = 2m\,\bar{u}(p')\gamma_5 u(p)$

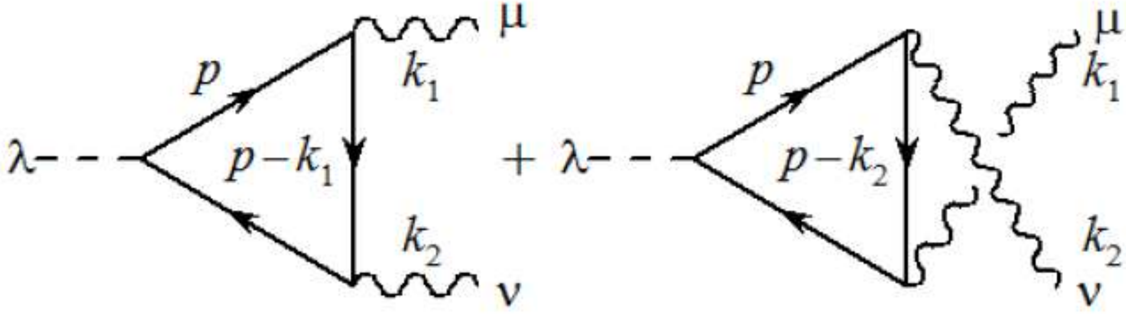
Axial current is conserved for massless fermions: chiral symmetry

It is not possible to maintain both symmetries when loop corrections are included. This is called: AXIAL ANOMALY



photons are bosons and they are not distinguishable hence amplitude has to be symmetrized

Naïve current conservation



$$q = k_1 + k_2$$

Skipping coupling constants (charges) the amplitude reads:

$$\begin{aligned} T_{\mu\nu\lambda} &= -i\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\nu\frac{i}{(\not p - \not k_1) - m}\gamma_\mu\right] \\ & \quad -i\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\mu\frac{i}{(\not p - \not k_2) - m}\gamma_\nu\right] \end{aligned}$$

Naively we expect:

$$k_1^\mu T_{\mu\nu\lambda} = k_2^\nu T_{\mu\nu\lambda} = 0 \qquad q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu}$$

Naïve current conservation

Vector current, first diagram:

$$k_1^\mu T_{\mu\nu\lambda} > \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\not{k}_1 \frac{i}{\not{p}-m}\right]$$

use trick:

$$\not{k}_1 = (\not{p}-m) - ((\not{p}-\not{k}_1)-m)$$

we get:

$$= i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\right] - i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{\not{p}-m}\right]$$

Naïve current conservation

Vector current, first diagram:

$$k_1^\mu T_{\mu\nu\lambda} > \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\not k_1 \frac{i}{\not p - m}\right]$$

use trick:

$$\not k_1 = (\not p - m) - ((\not p - \not k_1) - m)$$

we get:

$$= \ i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\right] - i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\nu \frac{i}{\not p - m}\right]$$

same trick with the second diagram gives

$$= \ i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m}\gamma_\nu \frac{i}{\not p - m}\right] - i\,\mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not k_2) - m}\gamma_\nu \frac{i}{\not p - m}\right]$$

Naïve current conservation

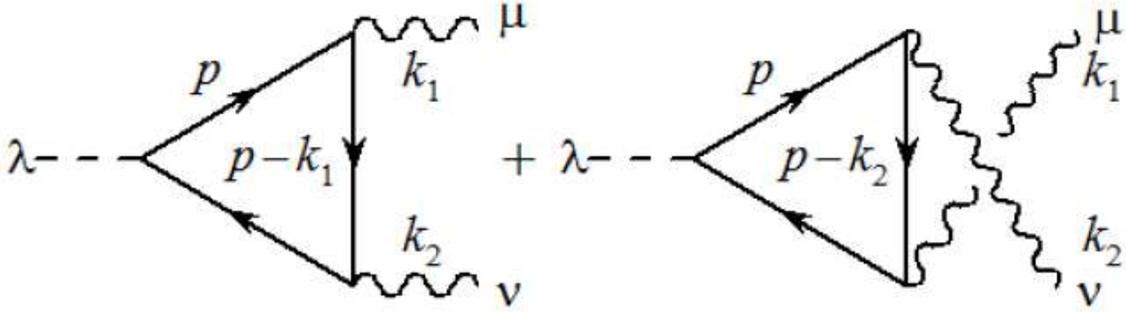
$$k_1^\mu T_{\mu\nu\lambda} \sim \int \frac{d^4p}{(2\pi)^4}$$

$$\left\{ \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{q})-m}\gamma_\nu \frac{i}{(\not{p}-\not{k}_1)-m}\right] - \mathrm{Tr}\left[\gamma_\lambda\gamma_5 \frac{i}{(\not{p}-\not{k}_2)-m}\gamma_\nu \frac{i}{\not{p}-m}\right]\right\}$$

change variable in the first integral $p \to p + k_1$

It seems we get zero

Naïve current conservation



$$q = k_1 + k_2$$

Skipping coupling constants (charges) the amplitude reads:

$$T_{\mu\nu\lambda} = -i\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma_\lambda\gamma_5\frac{i}{(\not{p}-\not{q})-m}\gamma_\nu\frac{i}{(\not{p}-\not{k}_1)-m}\gamma_\mu\right]$$
$$-i\int \frac{d^4p}{(2\pi)^4}\mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma_\lambda\gamma_5\frac{i}{(\not{p}-\not{q})-m}\gamma_\mu\frac{i}{(\not{p}-\not{k}_2)-m}\gamma_\nu\right]$$

Naively we expect:

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu}$$

Axial current

To calculate $q^\lambda T_{\mu\nu\lambda}$

we use the following trick:

$$
\begin{aligned}
\not{q}\gamma_5 &= -\gamma_5 \not{q} \\
&= \gamma_5 \left[(\not{p} - \not{q}) - m \right] - \gamma_5 \left[\not{p} - m \right] \\
&= \gamma_5 \left[(\not{p} - \not{q}) - m \right] + \left[\not{p} - m \right] \gamma_5 + 2m\gamma_5
\end{aligned}
$$

and for the first diagram we obtain

$$
\begin{aligned}
q^\lambda \left[\frac{i}{\not{p} - m} \gamma_\lambda \gamma_5 \frac{i}{(\not{p} - \not{q}) - m} \right] &= 2m \frac{i}{\not{p} - m} \gamma_5 \frac{i}{(\not{p} - \not{q}) - m} \\
&\quad + i \frac{i}{\not{p} - m} \gamma_5 + i\gamma_5 \frac{i}{(\not{p} - \not{q}) - m}
\end{aligned}
$$

Axial current

Sum from the two diagrams

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}$$

$$\begin{aligned}
&\Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \\
= &\int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p} - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m}\gamma_\mu + \gamma_5 \frac{i}{(\not{p} - \not{q}) - m}\gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m}\gamma_\mu\right] \\
+ &\int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p} - m}\gamma_5\gamma_\mu \frac{i}{(\not{p} - \not{k}_2) - m}\gamma_\nu + \gamma_5 \frac{i}{(\not{p} - \not{q}) - m}\gamma_\mu \frac{i}{(\not{p} - \not{k}_2) - m}\gamma_\nu\right]
\end{aligned}$$

Axial current

$$\Delta^{(1)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\gamma_\mu - \frac{i}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not q) - m}\gamma_\mu\right]$$

$$\Delta^{(2)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\mu \frac{i}{(\not p - \not k_2) - m}\gamma_\nu - \frac{i}{(\not p - \not k_1) - m}\gamma_5\gamma_\mu \frac{i}{(\not p - \not q) - m}\gamma_\nu\right]$$

The question is: are $\Delta^{(1,2)}_{\mu\nu}$ equal zero?

Changing variables $p \to p + k_2$

$p \to p + k_1$

seems to nullify $\Delta^{(1,2)}_{\mu\nu}$.

However, $\Delta^{(1,2)}_{\mu\nu} \sim \int^{\infty} dp p^3 \frac{1}{p^2} \sim \int^{\infty} dp p$ are UV divergent

Due to the minus sign the divergence is only linear

Mathematical diggression

Consider the integral that is naively zero:

$$\int_{-\infty}^{\infty} dx\,[f(x+a) - f(x)]$$

However, if

$$f(\pm\infty) \neq 0.$$

we can calculate this integral by Taylor expansion:

$$\int_{-\infty}^{\infty} dx\,[f(x+a) - f(x)] = a\,[f(\infty) - f(-\infty)] + \frac{a^2}{2}\,[f'(\infty) - f'(-\infty)] + \ldots$$

it may happen that $\neq 0$

Mathematical diggression

Consider Euclidean integral:

$$\Delta(\vec{a}) = \int d^n\vec{r}\,[f(\vec{r}+\vec{a}) - f(\vec{r})]$$

expand in a

$$= \int d^n\vec{r}\,\vec{a}\cdot\vec{\nabla} f(\vec{r}) + \ldots$$

apply Gauss theorem

$$= \vec{a}\cdot\vec{n}\,S_n(R)\,f(\vec{R})$$

where $\vec{n} = \frac{\vec{R}}{R}$ and $S_n(R)$ is a surface of the n sphere, R is regulator.

For even n

$$S_n(R) = \frac{2\pi^{n/2}}{(n/2-1)!}R^{n-1} = \begin{cases} 2\pi R & \text{for} \quad n=2 \\ 2\pi^2 R^3 & \text{for} \quad n=4 \end{cases}$$

Shift in full amplitude

$$\begin{aligned} T_{\mu\nu\lambda} &= -i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma_\lambda\gamma_5\frac{i}{(\not{p}-\not{q})-m}\gamma_\nu\frac{i}{(\not{p}-\not{k}_1)-m}\gamma_\mu\right] \\ &\quad -i\int\frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not{p}-m}\gamma_\lambda\gamma_5\frac{i}{(\not{p}-\not{q})-m}\gamma_\mu\frac{i}{(\not{p}-\not{k}_2)-m}\gamma_\nu\right] \end{aligned}$$

define shift vector $a = \alpha k_1 + (\alpha - \beta)k_2$

and amplitude difference:

$$\Delta_{\mu\nu\lambda}(a) = T_{\mu\nu\lambda}(p \to p + a) - T_{\mu\nu\lambda}$$

Strategy:

$$\begin{aligned} q^\lambda T_{\mu\nu l}(a) &= q^\lambda\left(T_{\mu\nu l}(a) - T_{\mu\nu l}(0)\right) + q^\lambda T_{\mu\nu l}(0) \\ &= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \end{aligned}$$

$$k_1^\mu T_{\mu\nu l}(a) = k_1^\mu\left(T_{\mu\nu l}(a) - T_{\mu\nu l}(0)\right) + k_1^\mu T_{\mu\nu l}(0)$$

chose a in a way that vector current is conserved
and see what comes out for the axial current

Shift in full amplitude

Calculate

(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}\left\{\mathrm{Tr}\left[\frac{1}{\not p+\not a-m}\gamma_\lambda\gamma_5\frac{1}{(\not p+\not a-\not q)-m}\gamma_\nu\frac{1}{(\not p+\not a-\not k_1)-m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not p-m}\gamma_\lambda\gamma_5\frac{1}{(\not p-\not q)-m}\gamma_\nu\frac{1}{(\not p-\not k_1)-m}\gamma_\mu\right]\right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}\left\{\mathrm{Tr}\left[\frac{1}{\not p + \not a - m}\gamma_\lambda\gamma_5\frac{1}{(\not p + \not a - \not q) - m}\gamma_\nu\frac{1}{(\not p + \not a - \not k_1) - m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5\frac{1}{(\not p - \not q) - m}\gamma_\nu\frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right]\right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma}\mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5\frac{1}{(\not p - \not q) - m}\gamma_\nu\frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right] + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} \left\{ \mathrm{Tr}\left[\frac{1}{\not p + \not a - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p + \not a - \not q) - m}\gamma_\nu \frac{1}{(\not p + \not a - \not k_1) - m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p - \not q) - m}\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right]\right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2) \,.$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} \mathrm{Tr}\left[\frac{1}{\not p - m}\gamma_\lambda\gamma_5 \frac{1}{(\not p - \not q) - m}\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right] + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2) \,.$$

large p limit

$$\frac{1}{p^6}\mathrm{Tr}\left[\not p\gamma_\lambda\gamma_5\not p\gamma_\nu\not p\gamma_\mu\right]$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}\left\{\mathrm{Tr}\left[\frac{1}{\not{p}+\not{a}-m}\gamma_\lambda\gamma_5\frac{1}{(\not{p}+\not{a}-\not{q})-m}\gamma_\nu\frac{1}{(\not{p}+\not{a}-\not{k}_1)-m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not{p}-m}\gamma_\lambda\gamma_5\frac{1}{(\not{p}-\not{q})-m}\gamma_\nu\frac{1}{(\not{p}-\not{k}_1)-m}\gamma_\mu\right]\right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}a^\sigma\frac{\partial}{\partial p^\sigma}\mathrm{Tr}\left[\frac{1}{\not{p}-m}\gamma_\lambda\gamma_5\frac{1}{(\not{p}-\not{q})-m}\gamma_\nu\frac{1}{(\not{p}-\not{k}_1)-m}\gamma_\mu\right] + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

large p limit

$$\frac{1}{p^6}\mathrm{Tr}\left[\not{p}\gamma_\lambda\gamma_5\not{p}\gamma_\nu\not{p}\gamma_\mu\right]$$

go to Euclidean
apply Gauss th.

$r_0 \to ir_0$

$d^4r = id^4\vec{r}$

$$\Delta_{\mu\nu\lambda}(a) = -\frac{i}{(2\pi)^4}2\pi^2a^\sigma \lim_{P\to\infty} P^3\frac{P_\sigma}{P}\mathrm{Tr}\left[\not{P}\gamma_\lambda\gamma_5\not{P}\gamma_\nu\not{P}\gamma_\mu\right]\frac{1}{P^6}$$

$$+ (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

Shift in full amplitude

Calculate
(all i's give -)

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4}\left\{\mathrm{Tr}\left[\frac{1}{\not{p}+\not{a}-m}\gamma_\lambda\gamma_5\frac{1}{(\not{p}+\not{a}-\not{q})-m}\gamma_\nu\frac{1}{(\not{p}+\not{a}-\not{k}_1)-m}\gamma_\mu\right] - \mathrm{Tr}\left[\frac{1}{\not{p}-m}\gamma_\lambda\gamma_5\frac{1}{(\not{p}-\not{q})-m}\gamma_\nu\frac{1}{(\not{p}-\not{k}_1)-m}\gamma_\mu\right]\right\} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

Expand in a

$$\Delta_{\mu\nu\lambda}(a) = -\int \frac{d^4p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma}\mathrm{Tr}\left[\frac{1}{\not{p}-m}\gamma_\lambda\gamma_5\frac{1}{(\not{p}-\not{q})-m}\gamma_\nu\frac{1}{(\not{p}-\not{k}_1)-m}\gamma_\mu\right] + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,.$$

large p limit

$$\frac{1}{p^6}\mathrm{Tr}\left[\not{p}\gamma_\lambda\gamma_5\not{p}\gamma_\nu\not{p}\gamma_\mu\right]$$

go to Euclidean
apply Gauss th.

$$\Delta_{\mu\nu\lambda}(a) = -\frac{i}{(2\pi)^4}2\pi^2 a^\sigma \lim_{P\to\infty} P^3\frac{P_\sigma}{P}\mathrm{Tr}\left[\not{P}\gamma_\lambda\gamma_5\not{P}\gamma_\nu\not{P}\gamma_\mu\right]\frac{1}{P^6} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

calculate Trace

$$\mathrm{Tr}\left[\not{P}\gamma_\lambda\gamma_5\not{P}\gamma_\nu\not{P}\gamma_\mu\right] = 4iP^2\varepsilon_{\alpha\mu\nu\lambda}P^\alpha$$

Remember that $\varepsilon_{\alpha\mu\nu\lambda} = -\varepsilon^{\alpha\mu\nu\lambda}$

Shift in full amplitude

We arrive at:

$$\Delta_{\mu\nu\lambda}(a) = \frac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha}\, a_\sigma \lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

Shift in full amplitude

We arrive at:
$$\Delta_{\mu\nu\lambda}(a) = \frac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha}\, a_\sigma \lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

take symmetric limit:
$$\lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} = \frac{1}{4} g^{\sigma\alpha}$$

recall: $a = \alpha k_1 + (\alpha - \beta) k_2$

Shift in full amplitude

We arrive at: $$\Delta_{\mu\nu\lambda}(a) = \frac{1}{(2\pi)^4} 8\pi^2 \varepsilon_{\mu\nu\lambda\alpha}\, a_\sigma \lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)$$

take symmetric limit: $$\lim_{P\to\infty} \frac{P^\sigma P^\alpha}{P^2} = \frac{1}{4} g^{\sigma\alpha}$$

recall: $a = \alpha k_1 + (\alpha - \beta) k_2$

Final result:
$$\begin{aligned}
\Delta_{\mu\nu\lambda}(a) &= \frac{1}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} a^\alpha + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2) \\
&= \frac{1}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(\alpha k_1^\alpha + (\alpha - \beta) k_2^\alpha - \alpha k_2^\alpha - (\alpha - \beta) k_1^\alpha\right) \\
&= \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha .
\end{aligned}$$

depends on β, there is an ambiguity, which we have to fix demanding that vector current is conserved.

Axial current, cont.

Recall:

$$q^\lambda T_{\mu\nu\lambda}(a) = q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0)$$
$$= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}$$

calculated   finite   needs to be computed

Axial current, cont.

Recall:

$$\begin{aligned} q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\ &= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \end{aligned}$$

calculated finite needs to be computed

Let's calculate

$$\Delta^{(1)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not{p} - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{k}_1) - m}\gamma_\mu - \frac{i}{(\not{p} - \not{k}_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not{p} - \not{q}) - m}\gamma_\mu\right]$$

Axial current, cont.

Recall:

$$\begin{aligned} q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\ &= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \end{aligned}$$

calculated finite needs to be computed

Let's calculate

$$\begin{aligned} \Delta^{(1)}_{\mu\nu} &= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\gamma_\mu - \frac{i}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not q) - m}\gamma_\mu\right] \\ &= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{1}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{1}{(\not p - \not k_2 - \not k_1) - m}\gamma_\mu - \frac{1}{\not p - m}\gamma_5\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right] \end{aligned}$$

We can use the same trick as previously $p \to p - k_2$ where $a = -k_2$

Axial current, cont.

Recall:

$$
\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\
&= q^\lambda \Delta_{\mu\nu\lambda}(a) + 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}
\end{aligned}
$$

calculated — finite — needs to be computed

Let's calculate

$$
\begin{aligned}
\Delta^{(1)}_{\mu\nu} &= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\gamma_\mu - \frac{i}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not q) - m}\gamma_\mu\right] \\
&= \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{1}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{1}{(\not p - \not k_2 - \not k_1) - m}\gamma_\mu - \frac{1}{\not p - m}\gamma_5\gamma_\nu \frac{1}{(\not p - \not k_1) - m}\gamma_\mu\right]
\end{aligned}
$$

We can use the same trick as previously $p \to p - k_2$ where $a = -k_2$

$$
\Delta^{(1)}_{\mu\nu} = -\frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho \lim_{P\to\infty} \frac{P_\rho}{P^2} \mathrm{Tr}\left[\not P \gamma_5 \gamma_\nu (\not P - \not k_1)\gamma_\mu\right]
$$

keep k_1, because Tr with 2 P's is zero

Axial current, cont.

$$\Delta^{(1)}_{\mu\nu} = -\frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho \lim_{P\to\infty} \frac{P_\rho}{P^2} \mathrm{Tr}\left[\not{P}\gamma_5\gamma_\nu(\not{P}-\not{k}_1)\gamma_\mu\right]$$

We have

$$\begin{aligned}
\Delta^{(1)}_{\mu\nu} &= \frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho k_1^\sigma \lim_{P\to\infty} \frac{P_\rho P^\alpha}{P^2} \mathrm{Tr}\left[\gamma_\alpha\gamma_5\gamma_\nu\gamma_\sigma\gamma_\mu\right] \\
&= \frac{1}{(2\pi)^4} 2i\pi^2 k_2^\rho k_1^\sigma \frac{1}{4}(-)\underbrace{\mathrm{Tr}\left[\gamma_5\gamma_\rho\gamma_\nu\gamma_\sigma\gamma_\mu\right]}_{4i\varepsilon_{\rho\nu\sigma\mu}} \\
&= -\frac{1}{8\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho.
\end{aligned}$$

We obtain $\Delta^{(2)}_{\mu\nu}$ by $\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2$, hence

$$\Delta^{(1)}_{\mu\nu} = \Delta^{(2)}_{\mu\nu}$$

Axial current, final

$$
\begin{aligned}
q^\lambda T_{\mu\nu\lambda}(a) &= q^\lambda \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + q^\lambda T_{\mu\nu\lambda}(0) \\
&= 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} + q^\lambda \Delta_{\mu\nu\lambda}(a) \\
&= 2mT_{\mu\nu} - \frac{1}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho + (k_1 + k_2)^\lambda \frac{\beta}{8\pi^2}\varepsilon_{\alpha\mu\nu\lambda}\,(k_1 - k_2)^\alpha \\
&= 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho
\end{aligned}
$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$\begin{aligned} k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha \\ &= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho. \end{aligned}$$

We ned the first piece

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}
$$

We ned the first piece

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda} &= -\int \frac{d^4p}{(2\pi)^4} \\
& \left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not a) - m} \gamma_\nu \frac{1}{(\not p - \not k_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2) - m} \gamma_\nu \frac{1}{\not p - m}\right]\right\}
\end{aligned}
$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}$$

We ned the first piece

$$\begin{aligned}
k_1^\mu T_{\mu\nu\lambda} &= -\int \frac{d^4 p}{(2\pi)^4} \\
&\quad \left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not q) - m} \gamma_\nu \frac{1}{(\not p - \not k_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2) - m} \gamma_\nu \frac{1}{\not p - m}\right] \right\}
\end{aligned}$$

⬇

$$\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2 - \not k_1)) - m} \gamma_\nu \frac{1}{(\not p - \not k_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2) - m} \gamma_\nu \frac{1}{\not p - m}\right] \right\}$$

Vector current

We shall use the same trick to calculate the divergence of a vecor current

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \left(T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)\right) + k_1^\mu T_{\mu\nu\lambda}(0) \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + k_1^\mu \frac{\beta}{8\pi^2} \varepsilon_{\alpha\mu\nu\lambda} \left(k_1 - k_2\right)^\alpha \\
&= k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}
$$

We ned the first piece

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda} &= -\int \frac{d^4 p}{(2\pi)^4} \\
& \left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not a) - m} \gamma_\nu \frac{1}{(\not p - \not k_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2) - m} \gamma_\nu \frac{1}{\not p - m}\right]\right\}
\end{aligned}
$$

$$
\left\{ \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2 - \not k_1)) - m} \gamma_\nu \frac{1}{(\not p - \not k_1) - m}\right] - \mathrm{Tr}\left[\gamma_\lambda \gamma_5 \frac{1}{(\not p - \not k_2) - m} \gamma_\nu \frac{1}{\not p - m}\right]\right\}
$$

$$
k_1^\mu T_{\mu\nu\lambda} = -\frac{1}{(2\pi)^4} 2i\pi^2 (-) k_1^\sigma \lim_{R\to\infty} \frac{P_\sigma}{P^2} \mathrm{Tr}\left[\gamma_\lambda \gamma_5 (\not P - \not k_2) \gamma_\nu \not P\right]
$$

Vector current

$$
\begin{aligned}
k_1^\mu T_{\mu\nu\lambda} &= -\frac{1}{(2\pi)^4} 2i\pi^2 (-) k_1^\sigma \lim_{R\to\infty} \frac{P_\sigma}{P^2} \operatorname{Tr}\left[\gamma_\lambda \gamma_5 (\not{P} - \not{k}_2)\gamma_\nu \not{P}\right] \\
&= -\frac{1}{8\pi^2} i \frac{1}{4} \operatorname{Tr}\left[\gamma_\lambda \gamma_5 \gamma_\rho \gamma_\nu \gamma_\sigma\right] k_1^\sigma k_2^\rho \\
&= \frac{1}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho .
\end{aligned}
$$

Recall

$$
k_1^\mu T_{\mu\nu\lambda}(a) = k_1^\mu T_{\mu\nu\lambda}(0) + \frac{\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho = \frac{1+\beta}{8\pi^2} \varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho
$$

We need to choose $\beta = -1$ to have vector current conserved!

Axial anomaly

Summarizing:

$$q^{\lambda} T_{\mu\nu\lambda}(a) \; = \; 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}k_1^{\sigma}k_2^{\rho}$$

$$k_1^{\mu} T_{\mu\nu\lambda}(a) = \frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho}k_1^{\sigma}k_2^{\rho}$$

Choose $\beta = -1$

$$q^{\lambda} T_{\mu\nu\lambda} = 2mT_{\mu\nu} - \frac{1}{2\pi^2}\varepsilon_{\mu\nu\sigma\rho}k_1^{\sigma}k_2^{\rho}$$

Axial current is anomalous

This can be translated to the configurations space

$$\partial^{\lambda} J_{\lambda}^{5}(x) = \frac{1}{(4\pi)^2}\varepsilon_{\mu\nu\sigma\rho}F^{\mu\nu}(x)F^{\sigma\rho}(x) + \mathcal{O}(m)$$

Axial anomaly

Summarizing:

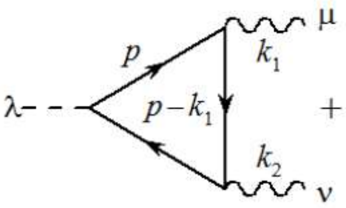
$$q^\lambda T_{\mu\nu\lambda}(a) \;=\; 2mT_{\mu\nu} - \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho$$

$$k_1^\mu T_{\mu\nu\lambda}(a) = \frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho$$

- Anomaly is mass independent
- Adler-Bardeen theorem (69): no higher order correctoons
- name: Adler-Bardeen-Jackiw anomaly
- Fujikawa (79) path integral formulation
- In non-Abelian case one can nullify anomaly Tr(…)=0

Choose $\beta = -1$

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} - \frac{1}{2\pi^2}\varepsilon_{\mu\nu\sigma\rho}\, k_1^\sigma k_2^\rho$$



Axial current is anomalous

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