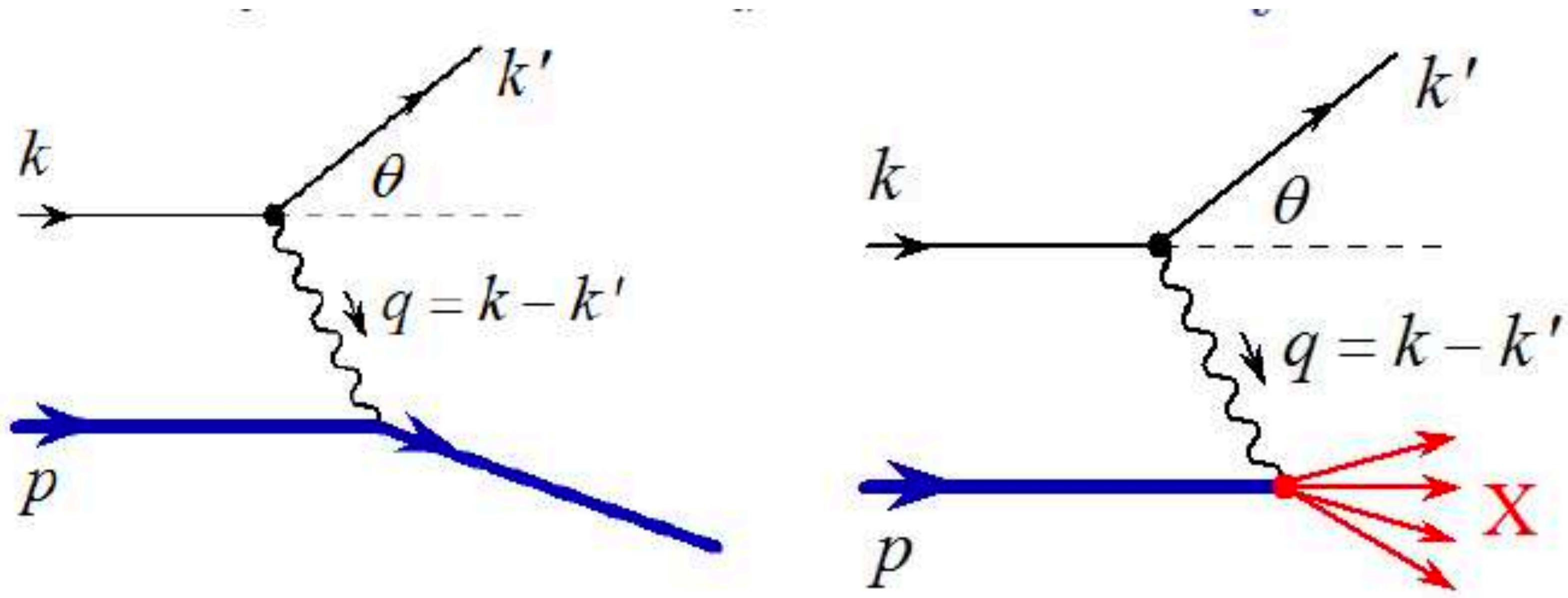
QCD Lecture 2

October 21

Deep Inelastic Scattering (DIS)



4-momentum transfer and energy transfer

$$q^2 = -2\omega\omega'(1-\cos\theta) = -4\omega\omega'\sin^2\frac{\theta}{2}, \qquad \nu = \omega - \omega'$$

on mass-shell condition for scattered proton (not present in the inelastic case):

$$\delta((p+q)^2 - M^2) = \delta(2M\nu - Q^2) = \frac{1}{2M}\delta\left(\nu - \frac{Q^2}{2M}\right)$$

Elastic cross-section:

$$\frac{d\sigma}{dQ^2} = \frac{\pi\alpha^2}{4\omega^2 \sin^4\frac{\theta}{2}} \int \frac{e_p^2}{\omega\omega'} \left\{ \frac{\mathcal{A}}{4} \cos^2\frac{\theta}{2} - \frac{\mathcal{B}}{2M^2} \sin^2\frac{\theta}{2} \right\} d\nu\, \delta\left(\nu - \frac{Q^2}{2M}\right)$$

$$= \frac{\pi\alpha^2}{4\omega^2 \sin^4\frac{\theta}{2}} \frac{e_p^2}{\omega\omega'} \left\{ \cos^2\frac{\theta}{2} + \frac{Q^2}{2M^2} \sin^2\frac{\theta}{2} \right\}.$$

Recall:

$$\frac{1}{4}\sum_{\text{pol}} |\mathcal{M}_{fi}|^2 = \frac{e_1^2 e_2^2}{(q^2)^2} L^{\nu\mu}(k,k')\, L_{\nu\mu}(p,p')$$

$$L_{\nu\mu}(p,q) = 4 \overset{A}{\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right)\left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)} + q^2 \overset{B}{\left(g_{\nu\mu} - \frac{q_\nu q_\mu}{q^2}\right)}$$

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$$= \frac{\pi\alpha^2}{4\omega^2 \sin^4\frac{\theta}{2}} \frac{e_p^2}{\omega\omega'} \left\{ \cos^2\frac{\theta}{2} + \frac{Q^2}{2M^2} \sin^2\frac{\theta}{2} \right\}.$$

Recall:

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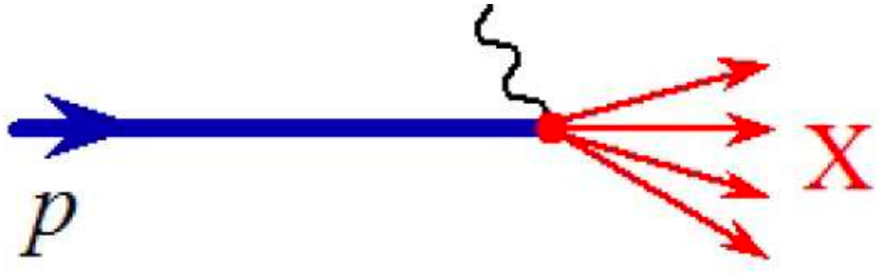
$$L_{\nu\mu}(p,q) = 4\overbrace{\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right)\left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)}^{A} + q^2 \overbrace{\left(g_{\nu\mu} - \frac{q_\nu q_\mu}{q^2}\right)}^{B}$$

Inelastic case:

1) v not fixed (X not mesured)

2) proton is not elementary



$$W_{\mu\nu}(p,q) = \underbrace{4W_2}_{\mathcal{A}} \left(p_\mu - \frac{p\cdot q}{q^2} q_\mu\right)\left(p_\nu - \frac{p\cdot q}{q^2} q_\nu\right) + \underbrace{4M^2 W_1}_{-\mathcal{B}} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)$$

Inelastic cross-section:

$$\frac{d\sigma}{dQ^2 d\nu} = \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{ \frac{\mathcal{A}}{4}\cos^2\frac{\theta}{2} - \frac{\mathcal{B}}{4M^2} 2\sin^2\frac{\theta}{2} \right\}$$
$$= \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{ W_2(Q^2,\nu)\cos^2\frac{\theta}{2} + 2W_1(Q^2,\nu)\sin^2\frac{\theta}{2} \right\}$$

Two unknown functions describing the proton structure: W_1 and W_2 depending on two independent variables: Q^2 and ν

Inelastic case:

1) ν not fixed (X not mesured)

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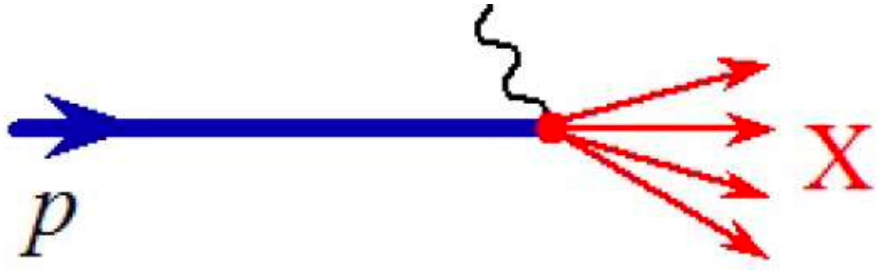

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Bjorken Scaling

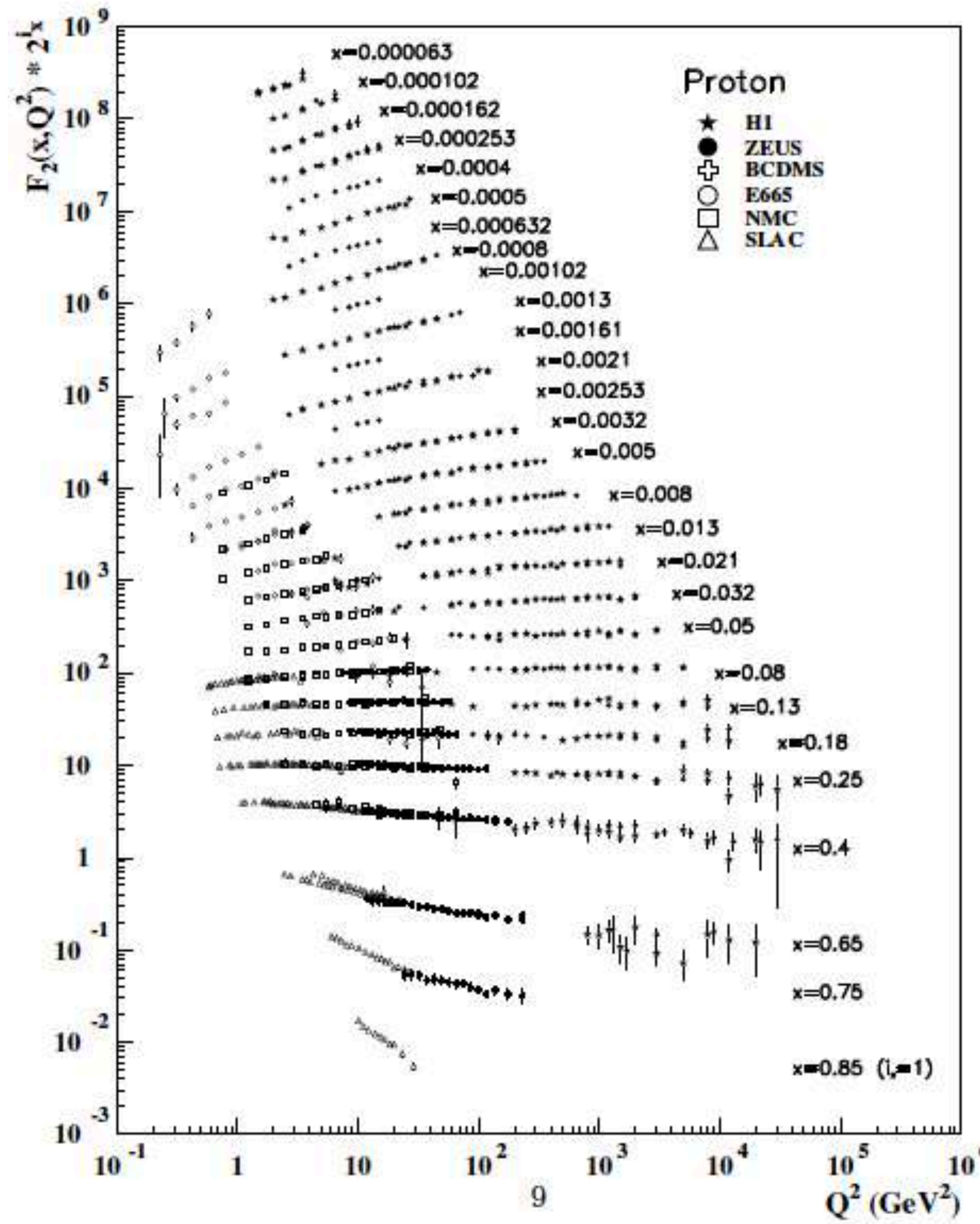
Bjorken limit:

$$Q^2, \nu \to \infty \qquad Q^2/\nu$$

$$MW_1(Q^2, \nu) = F_1(x)$$
$$\nu W_2(Q^2, \nu) = F_2(x)$$

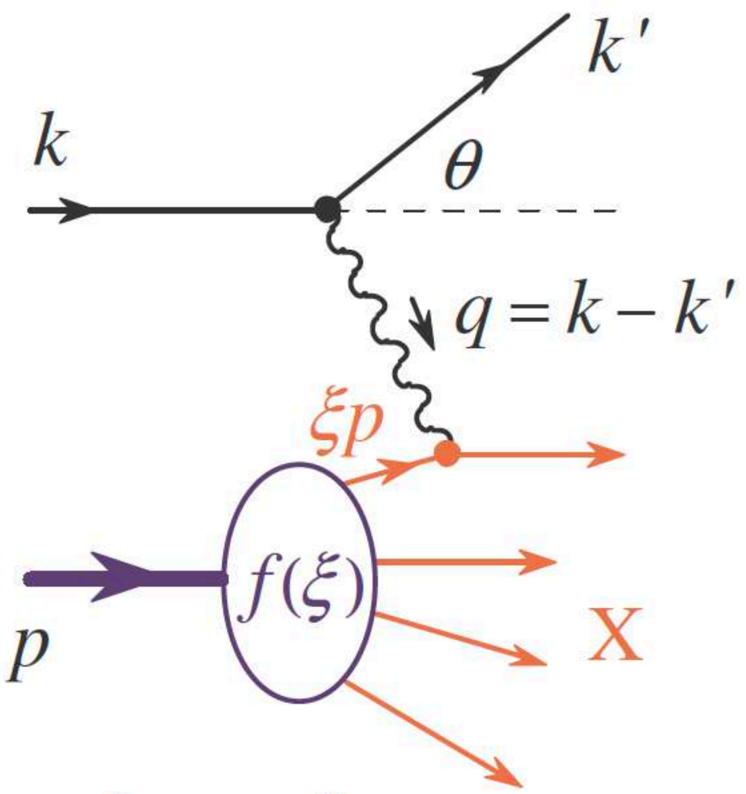
where:

$$x = \frac{Q^2}{2M\nu}$$



Feynman Parton Model

Inelastic scattering on proton
is a sum of elastic scattrings on partons
that are parallel to p
and carry momentum fraction ξ



In the proton rest frame we have to assume that parton mass is

$$m_\xi = \xi M$$

then the on-shell condition for the struck parton reads

$$(\xi p + q)^2 = m_\xi^2$$

$$\xi^2 M^2 + 2\xi M\nu - Q^2 = \xi^2 M^2 \;\to\; \xi = \frac{Q^2}{2M\nu} = x$$

ξ is the same as Bjorken x !

parton elastic cross-section with proton mas M replaced by $\xi_i M$
and proton charge replaced by parton charge e_i

$$\left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}} = \frac{\pi\alpha^2 e_i^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}} \left\{\cos^2\frac{\theta}{2} + \frac{Q^2}{4\xi_i^2 M^2} 2\sin^2\frac{\theta}{2}\right\} \delta\left(\nu - \frac{1}{\xi_i}\frac{Q^2}{2M}\right)$$

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multiply by probabilty of finding parton i in the proton,
sum over all partons and integrate over $d\xi_i$ and you get the inelastic cross-section on the proton

$$\frac{d\sigma}{dQ^2 d\nu} = \sum_i \int d\xi_i\, f_i(\xi_i) \left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}}$$

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multiply by probabilty of finding parton i in the proton,
sum over all partons and integrate over $d\xi_i$ and you get the inelastic cross-section on the proton
expresed in terms of the Bjorken functions $W_{1,2}$

$$\frac{d\sigma}{dQ^2 d\nu} = \sum_i \int d\xi_i\, f_i(\xi_i) \left.\frac{d\sigma_i}{dQ^2 d\nu}\right|_{\text{parton}} = \frac{\pi\alpha^2}{4\omega^3\omega' \sin^4\frac{\theta}{2}}\left\{W_2 \cos^2\frac{\theta}{2} + 2W_1 \sin^2\frac{\theta}{2}\right\}$$

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we can now immediately calculate $W_{1,2}$ in terms of $f(\xi)$

$$W_2 = \sum_i e_i^2 \int d\xi\, f_i(\xi)\delta\left(\nu - \nu\frac{x}{\xi}\right) = \sum_i e_i^2 \int d\xi\, f_i(\xi)\frac{\xi^2}{\nu x}\delta(\xi - x) = \frac{1}{\nu}\sum_i e_i^2 x\, f_i(x)$$

$$W_1 = \sum_i e_i^2 \int d\xi\, f_i(\xi)\frac{Q^2}{4\xi^2 M^2}\frac{\xi^2}{\nu x}\delta(\xi - x) = \frac{1}{2M}\sum_i e_i^2\, f_i(x). \qquad x = \frac{Q^2}{2M\nu}$$

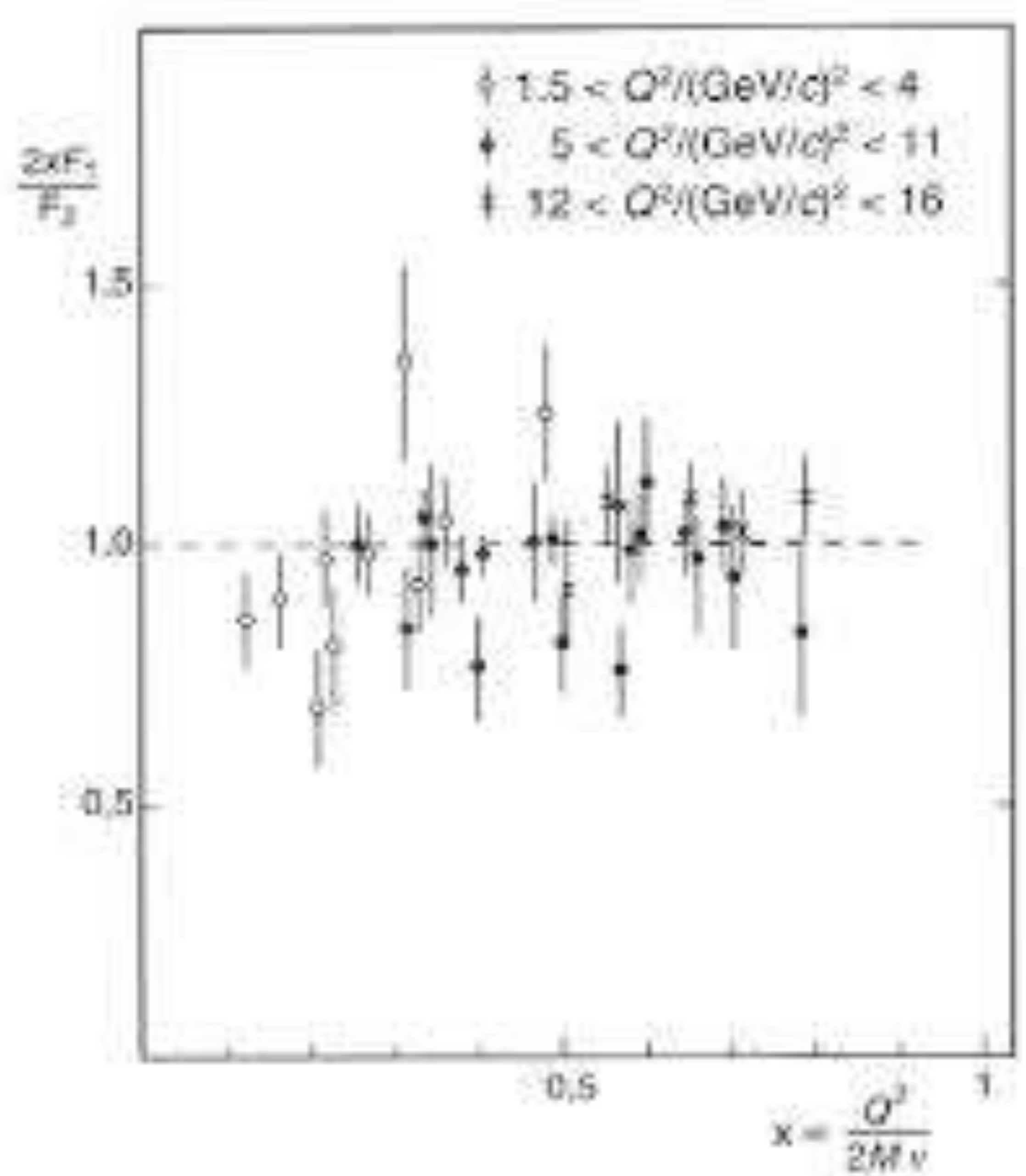
Bjorken Scaling vs. Parton Model

$$F_2(x) = \nu W_2 = x \sum_i e_i^2 f_i(x)$$

$$F_1(x) = M W_1 = \frac{1}{2} \sum_i e_i^2 f_i(x)$$

$$F_2(x) = 2x F_1(x)$$

in parton model structure fubctions are related: Callan-Gross relation



Quarks as Partons

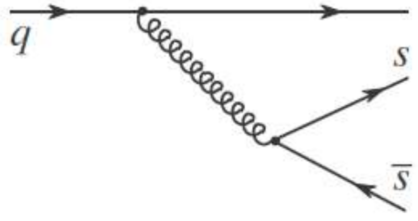
$$F_2^{\mathrm{p}}(x) = \frac{4}{9}x\left[u_{\mathrm{p}}(x) + \overline{u}_{\mathrm{p}}(x)\right] + \frac{1}{9}x\left[d_{\mathrm{p}}(x) + \overline{d}_{\mathrm{p}}(x) + s_{\mathrm{p}}(x) + \overline{s}_{\mathrm{p}}(x)\right]$$

$$F_2^{\mathrm{n}}(x) = \frac{4}{9}x\left[u_{\mathrm{n}}(x) + \overline{u}_{\mathrm{n}}(x)\right] + \frac{1}{9}x\left[d_{\mathrm{n}}(x) + \overline{d}_{\mathrm{n}}(x) + s_{\mathrm{n}}(x) + \overline{s}_{\mathrm{n}}(x)\right]$$

assuming isospin symmetry:

$$u_{\mathrm{p}} = d_{\mathrm{n}} = u, \quad d_{\mathrm{p}} = u_{\mathrm{n}} = d, \quad s_{\mathrm{p}} = s_{\mathrm{n}} = s$$

no strangeness in the nucleon: $\int dx(s(x) - \overline{s}(x)) = 0$



Quarks as Partons

proton and neutron charges

$$q_{\mathrm{p}} = \int dx \left[\frac{2}{3}(u(x) - \overline{u}(x)) - \frac{1}{3}(d(x) - \overline{d}(x)) - \frac{1}{3}(s(x) - \overline{s}(x))\right] = 1$$

$$\updownarrow = 0$$

$$q_{\mathrm{n}} = \int dx \left[\frac{2}{3}(d(x) - \overline{d}(x)) - \frac{1}{3}(u(x) - \overline{u}(x)) - \frac{1}{3}(s(x) - \overline{s}(x))\right] = 0$$

imply constraints on the parton distributions (PDF's):

$$\int dx(u(x) - \overline{u}(x)) = 2, \quad \int dx(d(x) - \overline{d}(x)) = 1, \quad \int dx(s(x) - \overline{s}(x)) = 0$$

valence and sea quarks: $u = u_v + q_s, \quad d = d_v + q_s, \quad \overline{u} = \overline{d} = \overline{s} = s = q_s$

total momenum – for typical parametrizations

$$\int dx\, x(u(x) + \overline{u}(x) + d(x) + \overline{d}(x) + s(x) + \overline{s}(x)) = 1 - \varepsilon$$

$\varepsilon \sim 45\%$

there must be other partons that do not inteact electromagnetically: gluons

Quantum Chromo Dynamics

$$\Psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix}$$

Gauge theory based on SU(3) group

$$\Psi(x) \to \Psi'(x) = U(x)\Psi(x) \qquad U(x) = e^{-i\theta_m(x)T^m} \qquad (m = 1, 2, \ldots N^2 - 1)$$

covariant derivative

$$D_\mu = \partial_\mu + igT^m A_\mu^m(x) = \partial_\mu + ig\boldsymbol{A}_\mu(x)$$

transforms as

$$D'_\mu = U(x)D_\mu U^\dagger(x) \longrightarrow$$

$$\longrightarrow \boldsymbol{A}'_\mu(x) = U(x)\boldsymbol{A}_\mu(x)U^\dagger(x) - \frac{i}{g}U(x)\partial_\mu U^\dagger(x)$$

SU(N) group

in fundamental representation generators are given as $N \times N$ hermitean matrices that satisfy commutation relations

$$[T_m, T_n] = i f_{mnl} T_l$$

f_{mnl} are totally antisymmetric tensors known as structure constants. To define the group we either give explicit form of the generators or a complete set of structure constants.

Examples:
SU(2)
Pauli matrices

$$T^i = \frac{1}{2}\tau^i$$

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Normalization:

$$\mathrm{Tr}(T_m T_n) = \frac{1}{2}\delta_{mn}$$

SU(N) group

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Examples:
SU(3)
Gell-Mann matrices

$$T^i = \frac{1}{2}\lambda^m$$

$$\lambda^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda^2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\lambda^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda^5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix},$$

$$\lambda^6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \lambda^7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix},$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Conjugated fundamental rep.

obviously, there are infintely many matrix representations related by the unitary transformation

$$T'_n = U^\dagger T_n U$$

let’s complex conjugate the commutation relation

$$[T_m, T_n] = i f_{mnl} T_l$$

and multiply all generators by minus

$$[-T^*_m, -T^*_n] = i f_{mnl} \left(-T^*_l\right)$$

we have constructed conjugated representation $T'_n = -T^*_n$ satysfying commutation relation

is this representation unitary equivalent to the fundamental one?

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is this representation unitary equivalent to the fundamental one?

SU(2) – yes
SU(3) and higher – no

complication

$$\tau_i \tau_j = \delta_{ij} + i\varepsilon_{ijk}\tau_k,$$

$$\lambda_a \lambda_b = \frac{2}{3}\delta_{ab} + i f_{abc}\lambda_c + d_{abc}\lambda_c$$

therefore quarks and antiquarks are different objects

Adjoint representation

it follows from the Jacobi identity

$$[T_m, [T_n, T_l]] + [T_n, [T_l, T_m]] + [T_l, [T_m, T_n]] = 0$$

that

$$f_{nlk} f_{kmr} + f_{lmk} f_{knr} + f_{mnk} f_{klr} = 0$$

this relation can be writen in terms of $(N^2\text{-}1) \times (N^2\text{-}1)$ matrices defined as

$$\left(T_l^{\text{adj}}\right)_{mn} = -i f_{lmn}$$

in the following way

$$[T_m, T_n] = i f_{mnl} T_l$$

which means that T_l^{adj} are SU(3) generators, they form adjoint representation
note that

$$-T_l^{\text{adj}\,*} = T_l^{\text{adj}}$$

so adjoint representation is self-conjugated (real)

Adjoint representation

consider vector in the adjoint representation $A = (a^1, \ldots, a^{N^2-1})$

which transforms as $A' = U^{\rm adj} A \to a'^m = a^m - \theta^l f_{lmn} a^n + \ldots .$

because $U(x) = e^{-i\theta_m(x)T^m}$ and $\left(T_l^{\rm adj}\right)_{mn} = -i f_{lmn}$

one can write this transformation differently, defining $\boldsymbol{A} = \sum_{n=1}^{N^2-1} a^n T_n$

then $\boldsymbol{A}' = U \boldsymbol{A} U^\dagger$

leads to

$$\begin{aligned} a'^m T_m &= (1 - i\,\theta^n T_n + \ldots) a^m T_m (1 + i\,\theta^n T_n + \ldots) \\ &= a^m T_m - i\,\theta^n \left[T_n, T_m\right] a^m \\ &= a^m T_m + \theta^n\, f_{nmk} T_k\, a^m \\ &= \left(a^m - \theta^l\, f_{lmn}\, a^n\right)\, T_m, \end{aligned}$$

Adjoint representation

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leads to

$$
\begin{aligned}
a'^m T_m &= (1 - i\,\theta^n T_n + \ldots) a^m T_m (1 + i\,\theta^n T_n + \ldots) \\
&= a^m T_m - i\,\theta^n \left[T_n, T_m\right] a^m \\
&= a^m T_m + \theta^n f_{nmk} T_k\, a^m \\
&= \left(a^m - \theta^l f_{lmn}\, a^n\right) T_m,
\end{aligned}
$$

gauge fields transform according to the adjoint representation of SU(N)

QED vs.QCD

field tensor in QED $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

can be expressed in terms of covariant derivatives, because the the field is Abelian:

$$F^{\mu\nu} = D^\mu A^\nu - D^\nu A^\mu = (\partial^\mu + iq\underline{A^\mu)\, A^\nu} - (\partial^\nu + iq\underline{A^\nu)\, A^\mu}$$

this can be generalized to the non Abelian case where the commutator does not vanish

$$\boldsymbol{F}_{\mu\nu} = D_\mu \boldsymbol{A}_\nu - D_\nu \boldsymbol{A}_\mu = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + ig\left[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu\right]$$

in order to find transformaion law, we have first to prove that

$$\boldsymbol{F}_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu]$$

commutator is in principle an operator
and the field tensor is a function!

because

$$D'_\mu = U(x) D_\mu U^\dagger(x)$$

we have

$$\boldsymbol{F}'_{\mu\nu} = U(x) \boldsymbol{F}_{\mu\nu} U^\dagger(x)$$

QCD Lagrangian

gauge boson part (yang-Mills)

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\,\mathrm{Tr}(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}) = -\frac{1}{4}\sum_m F^m_{\mu\nu}F^{m\,\mu\nu}$$

in QED $\quad (\partial_\mu A_\nu - \partial_\nu A_\mu)^2$

in QCD $\quad (\partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + ig[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu])^2$

QCD lagrangian contains interactions!
gluons interact with themselves, they carry adjoint color charge

QCD Lagrangian

gauge boson part (yang-Mills)

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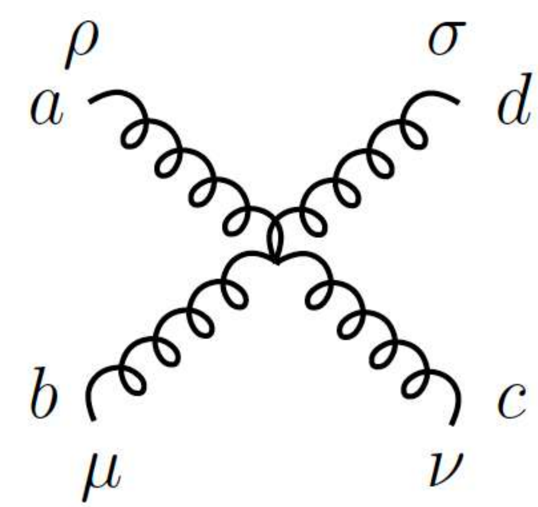
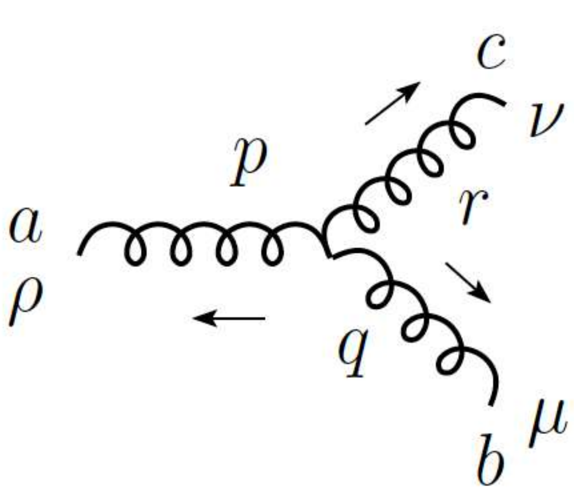


QCD lagrangian contains interactions!
gluons interact with themselves, they carry adjoint color charge

$$-ig_s^2 f^{abe} f^{cde} (g_{\rho\nu}g_{\mu\sigma} - g_{\rho\sigma}g_{\mu\nu})$$
$$-ig_s^2 f^{ace} f^{bde} (g_{\rho\mu}g_{\nu\sigma} - g_{\rho\sigma}g_{\mu\nu})$$
$$-ig_s^2 f^{ade} f^{cbe} (g_{\rho\nu}g_{\mu\sigma} - g_{\rho\mu}g_{\sigma\nu})$$



$$-g_s f^{abc}\left[(p-q)_\nu\, g_{\rho\mu} + (q-r)_\rho\, g_{\mu\nu} + (r-p)_\mu\, g_{\nu\rho}\right]$$

QCD Lagrangian

gauge boson part (yang-Mills)

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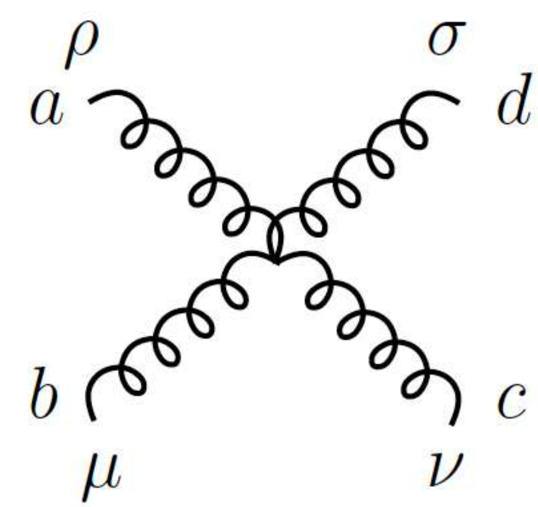

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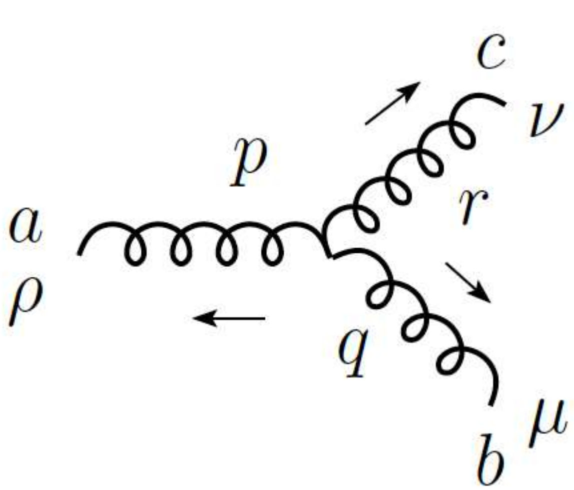
gluons interact with themselves, they carry adjoint color charge

$$\begin{aligned}
&-ig_s^2 f^{abe} f^{cde} (g_{\rho\nu}g_{\mu\sigma} - g_{\rho\sigma}g_{\mu\nu})\\
&-ig_s^2 f^{ace} f^{bde} (g_{\rho\mu}g_{\nu\sigma} - g_{\rho\sigma}g_{\mu\nu})\\
&-ig_s^2 f^{ade} f^{cbe} (g_{\rho\nu}g_{\mu\sigma} - g_{\rho\mu}g_{\sigma\nu})
\end{aligned}$$

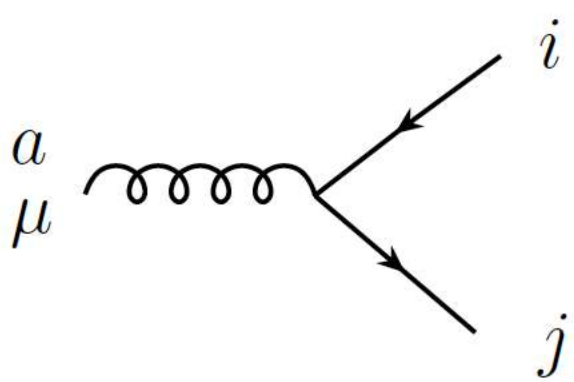


$$-g_s f^{abc}\,[(p-q)_\nu\, g_{\rho\mu} + (q-r)_\rho\, g_{\mu\nu} + (r-p)_\mu\, g_{\nu\rho}]$$

Full QCD lagrangian

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr}\left[\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}\right] + \sum_{f=1}^{6}\left[\overline{q}_f\, i\gamma^\mu D_\mu q_f - m_f \overline{q}_f\, q_f\right]$$

quarks interact via covariant derivative



$$ig_s\gamma_\mu T^a_{ji}$$

propagators:

$$iS_F(p) = \cdot\; i\,\delta_{ij}\frac{(\not{k}+m)}{k^2-m^2+i\epsilon}$$

gauge choice!

$$iD_F(p)_{\mu\nu} = \frac{-i\,\delta_{ab}}{k^2+i\epsilon}\left[g_{\mu\nu}-(1-\eta)\frac{k_\mu k_\nu}{k^2}\right]$$

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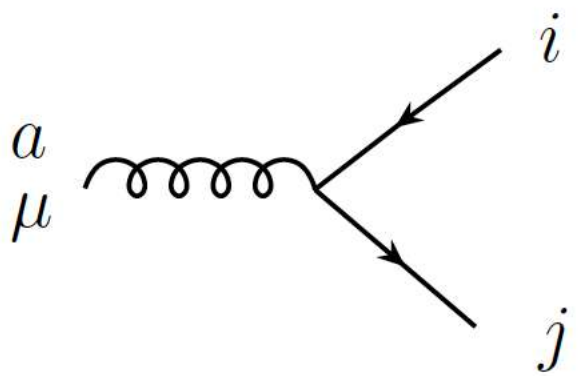

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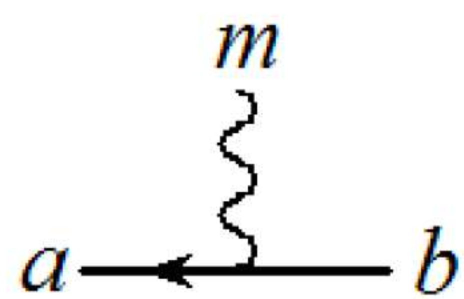
Color factors

each Feynman diagram is a product of a momentum-Dirac structure (like in QED) and a color factor

to calculate color factors it is very practical to use the graphical notation

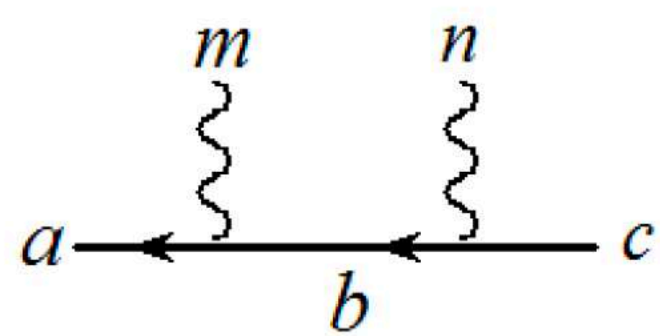
fundamental geneator:

$m,n = 1,2,\ldots N^2\text{-}1, \quad a,b = 1,2,\ldots N$



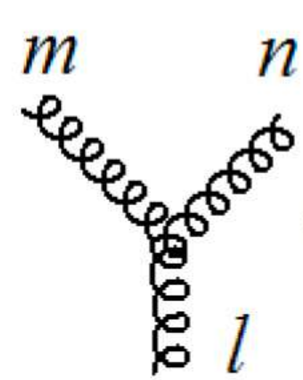
$= T^m_{ab}$

multiplication:



$= T^m_{ab}\, T^n_{bc}$

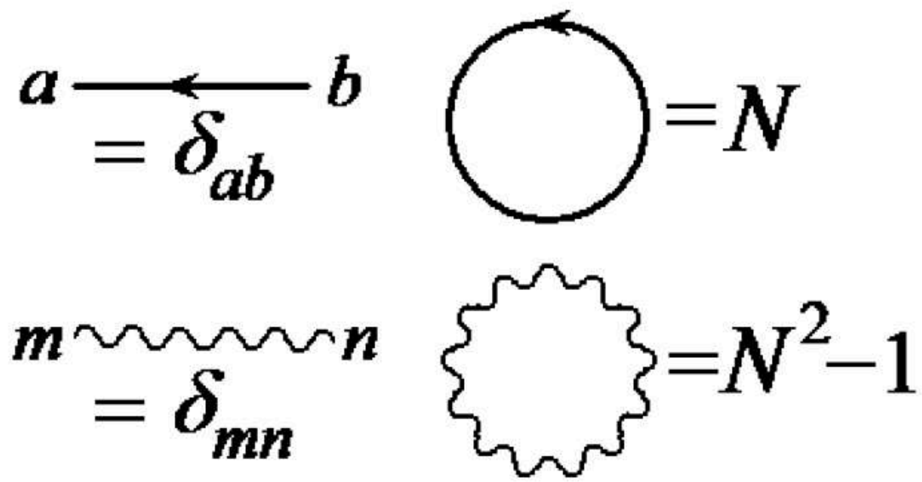
adjoint generator:



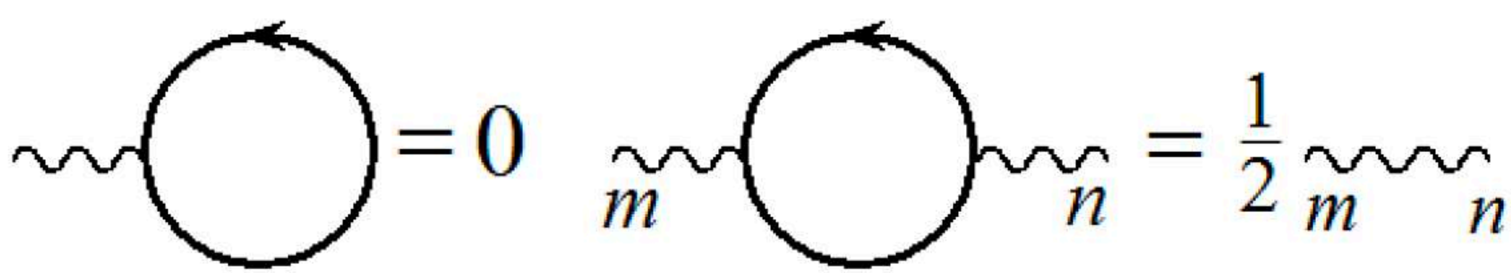
$= if_{mnl} = -if_{nml}$

Color factors

Kroneker deltas and traces:



generators are tracless and dormalized to 1/2

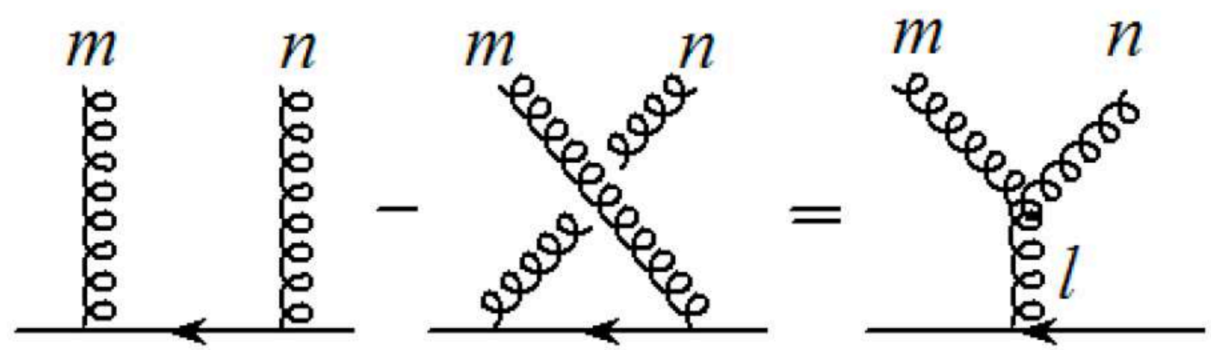


$$\mathrm{Tr}(T_m T_n) = \frac{1}{2}\delta_{mn}$$

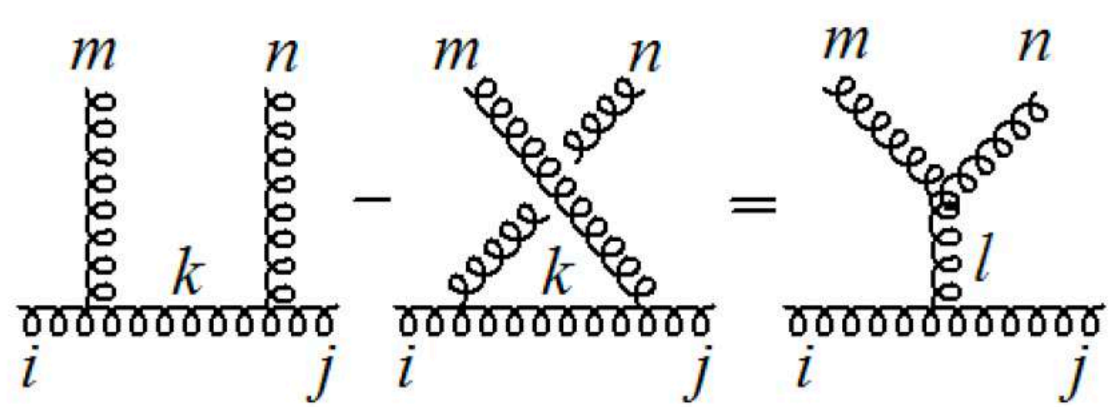
Color factors

commutation relations: $[T_m, T_n] = i f_{mnl} T_l$

fundamental:



adjoint:



Color factors

Example:

Casimir operator for the fundamental representation

quadratic Casimir operator is the sum over all generators squared
and it is proportional to unity multiplied by a number,
which is simply called "Casimir"

$$\sum_n (T^n)^2 = C_F \, \mathbf{1}$$

In SU(2) for any representation of spin *s* it is equal to

$$\sum_n \hat{S}_n^2 = s(s+1)\, \mathbf{1}$$

Color factors

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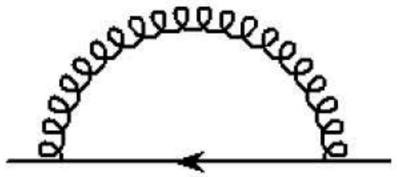
Casimir operator for the fundamental representation

$$\sum_n (T^n)^2 = C_F\,\mathbf{1}$$

$\Sigma_n (T^n)^2 =$  $= C_F$ 

Color factors

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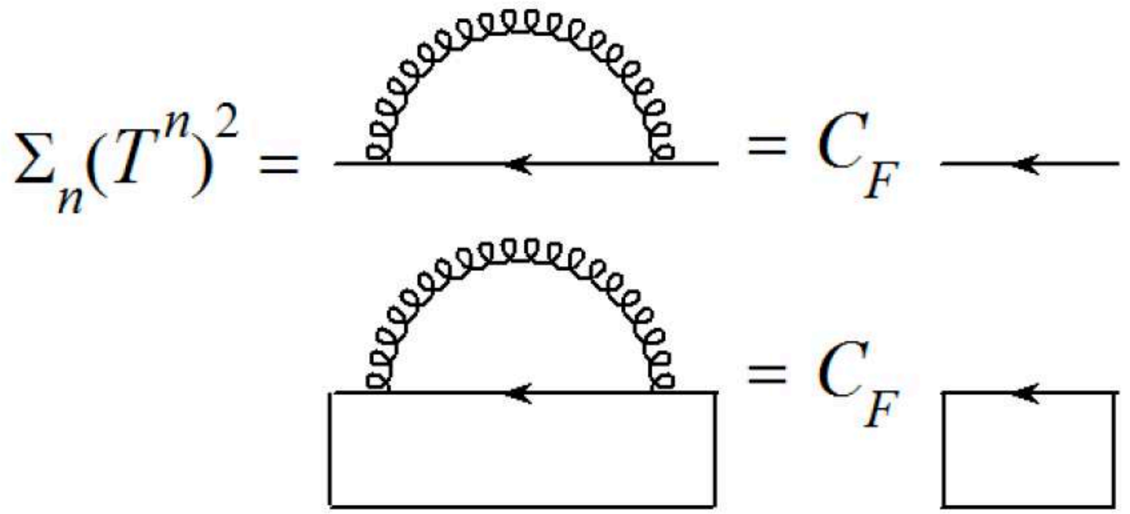
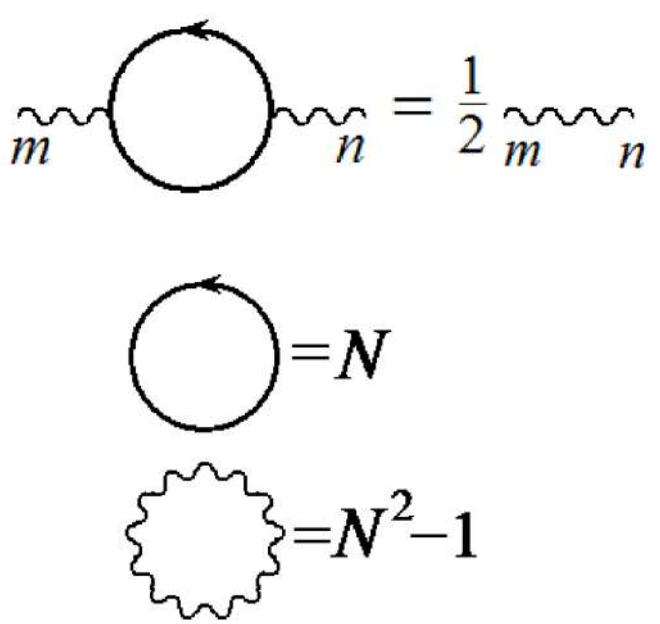


contract fermion line:

use:



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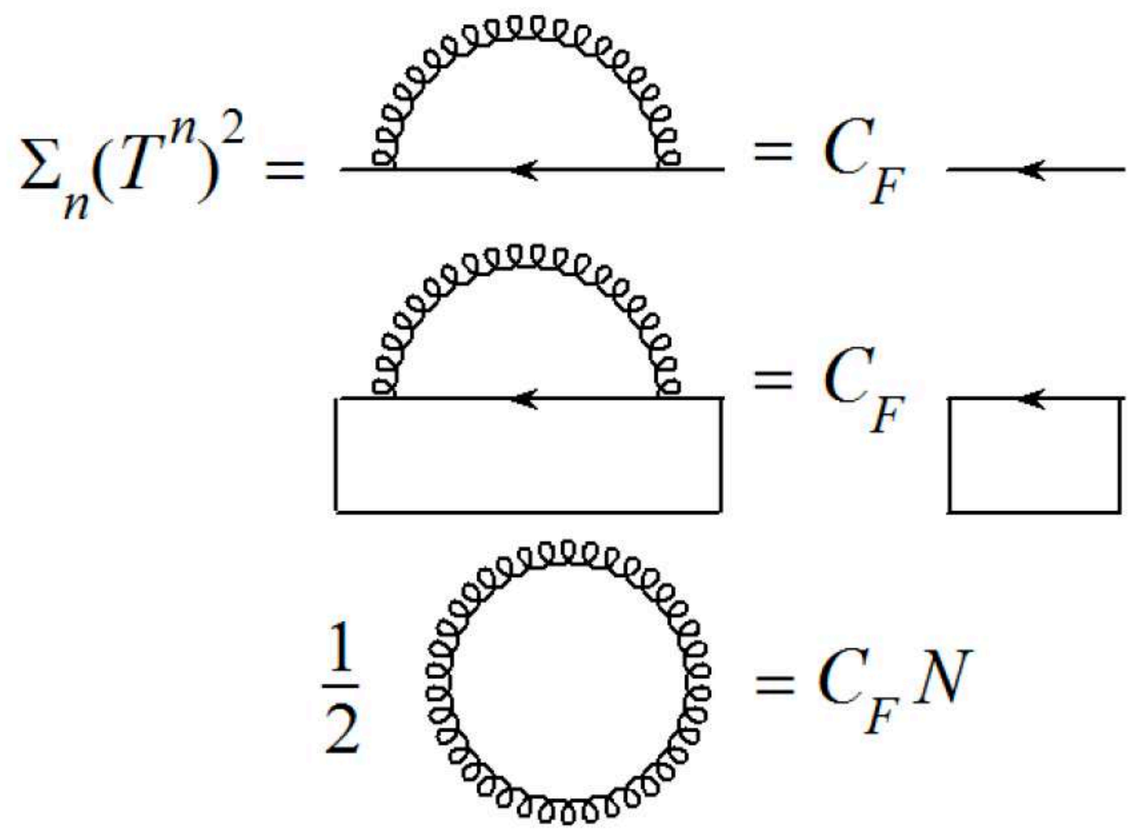

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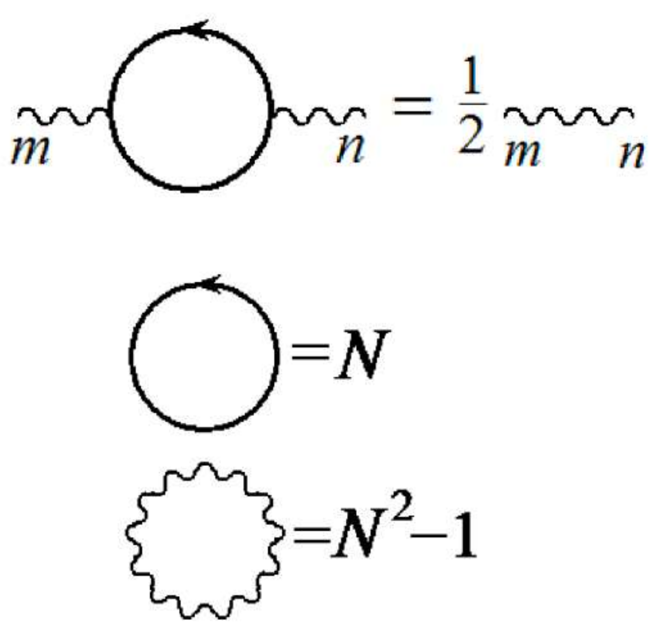

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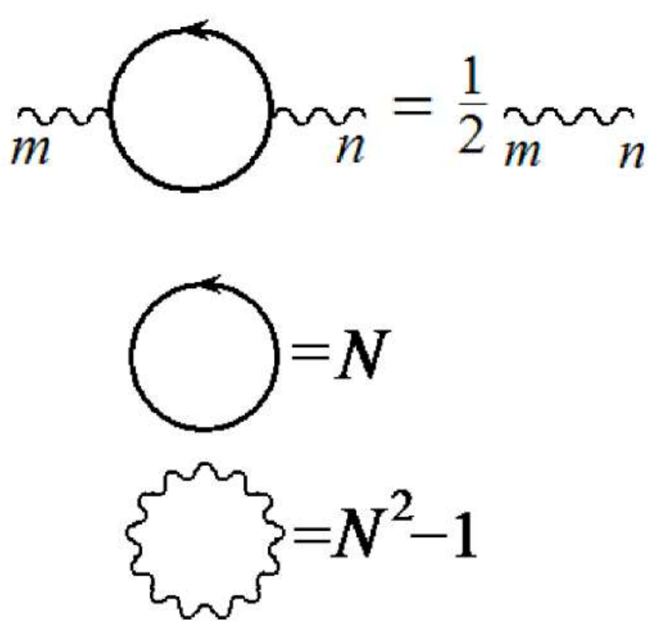
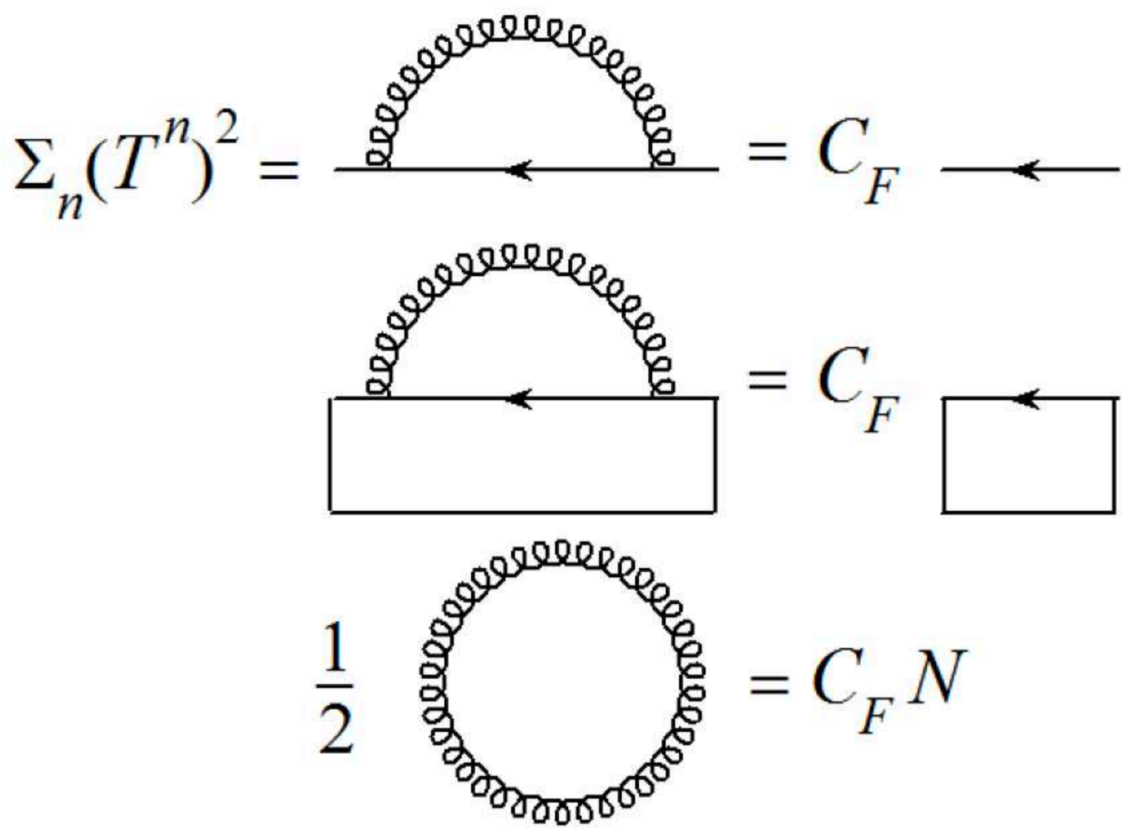
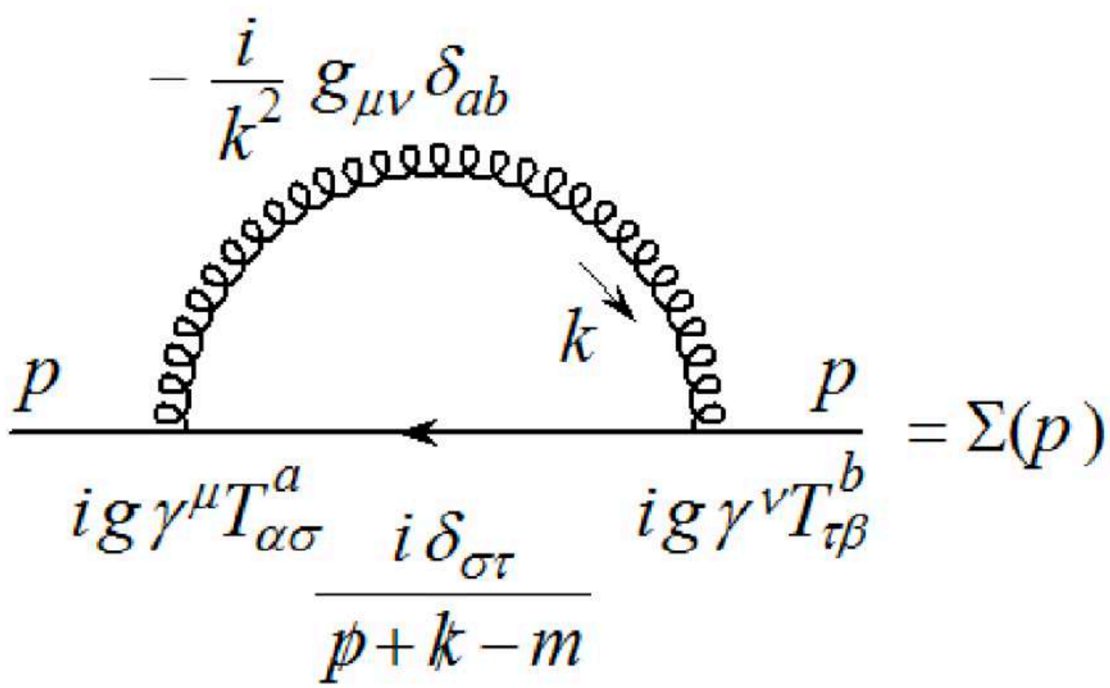




$$C_F = \frac{N^2-1}{2N} = \begin{cases} \frac{3}{4} & \text{SU(2)} \\ \frac{4}{3} & \text{SU(3)} \end{cases}$$

Renormalization

In quantum field theory loop diagrams have infinite integrals. We shall discuss this problem on the example of fermion self-energy in Feynman gauge.



$$\Sigma(p) = -g^2 C_F \delta_{\alpha\beta} \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not p + \not k + m)\gamma_\mu}{[(p+k)^2 - m^2]\; k^2}$$

This integral is logarithmically divergent for $k \longrightarrow$ infinity

We hve to first *regularize* it, so that we are dealing with finite quantities, and then we shall remove regulator. There are many ways to regularize the theory, we shall choose dimensional regularization

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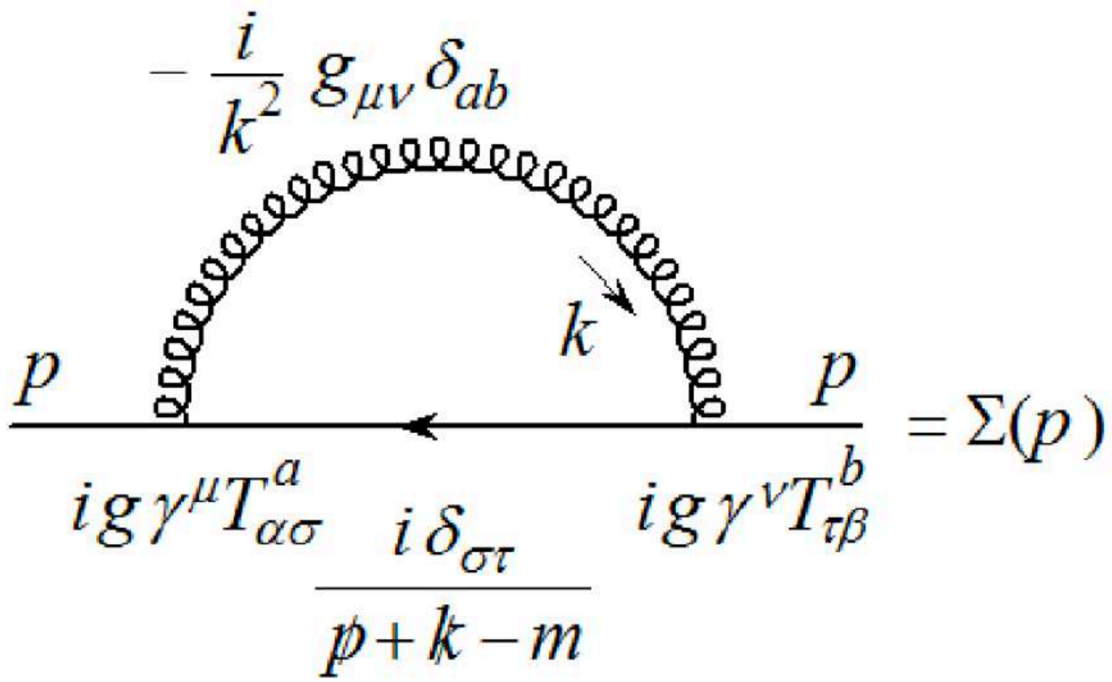

$$\Sigma(p) = -g^2 C_F \delta_{\alpha\beta} \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not p + \not k + m)\gamma_\mu}{[(p+k)^2 - m^2]\; k^2} \qquad \int \frac{d^4k}{(2\pi)^4}\frac{\not k}{(k^2)^2} = 0$$

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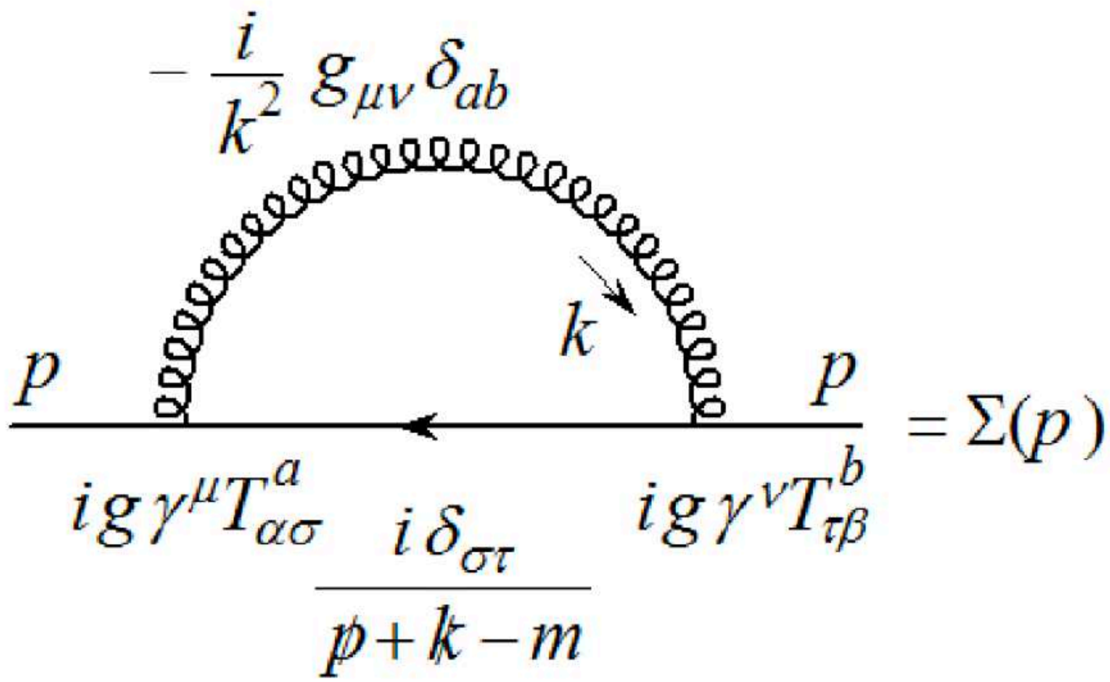

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$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} = \int \frac{k^3 dk d\Omega_4}{(2\pi)^4} \frac{1}{k^4} = \infty$$

This integral is logarithmically divergent for $k \longrightarrow$ infinity

We hve to first *regularize* it, so that we are dealing with finite quantities, and then we shall remove regulator. There are many ways to regularize the theory, we shall choose dimensional regularization

Dimensional regularization

$$4 \to d = 4 - 2\varepsilon$$

$$\Sigma(p) = -g^2 \mu^{4-d} C_F \delta_{\alpha\beta} \int \frac{d^d k}{(2\pi)^d} \frac{\gamma^\mu(\not{p} + \not{k} + m)\gamma_\mu}{[(p+k)^2 - m^2]\; k^2}$$

We want to keep the same dimensionality of Σ and g in any number of physical dimensions. We therefore introduce a dimensionfull parameter μ to correct for this.

We will extend Dirac algebra by simply using $g_{\mu\nu} g^{\mu\nu} = d$

It can be shown that we can treat Dirac bispinors as 4-dimensional.

Dimensional regularization preserves gauge invarince, but has problems in theories with γ_5. This is not the case of QCD.

In the following we shall keep $m = 0$.

Dirac algebra

We need to calculate

$$\gamma^\mu(\not{p}+\not{k})\gamma_\mu$$

with the help of the anticommutation rule: $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$

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$$\gamma^\mu(\not{p}+\not{k})\gamma_\mu \quad = \quad g_{\mu\nu}\gamma^\mu\gamma^\tau\gamma^\nu(p+k)_\tau$$

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commute γ^ν

$$\begin{aligned}\gamma^\mu (\not{p} + \not{k}) \gamma_\mu &= g_{\mu\nu} \gamma^\mu \gamma^\tau \gamma^\nu (p+k)_\tau \\ &= g_{\mu\nu} \gamma^\mu \left(2g^{\tau\nu} - \gamma^\nu \gamma^\tau\right) (p+k)_\tau\end{aligned}$$

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$$\begin{aligned} \gamma^\mu (\not{p} + \not{k}) \gamma_\mu &= g_{\mu\nu} \gamma^\mu \gamma^\tau \gamma^\nu (p+k)_\tau \\ &= g_{\mu\nu} \gamma^\mu \left(2g^{\tau\nu} - \gamma^\nu \gamma^\tau\right) (p+k)_\tau \\ &= 2(\not{p} + \not{k}) - d(\not{p} + \not{k}) \end{aligned}$$

$$g_{\mu\nu} \gamma^\mu \gamma^\nu = \frac{1}{2} g_{\mu\nu} \{\gamma^\mu, \gamma^\nu\} = g_{\mu\nu} g^{\mu\nu} = d$$

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use

$d = 4 - 2\varepsilon$

$$\begin{aligned} \gamma^\mu (\not{p} + \not{k}) \gamma_\mu &= g_{\mu\nu} \gamma^\mu \gamma^\tau \gamma^\nu (p+k)_\tau \\ &= g_{\mu\nu} \gamma^\mu \left(2g^{\tau\nu} - \gamma^\nu \gamma^\tau\right) (p+k)_\tau \\ &= 2(\not{p} + \not{k}) - d(\not{p} + \not{k}) \\ &= -2(1-\varepsilon)(\not{p} + \not{k}), \end{aligned}$$

$$\Sigma(p) = 2(1-\varepsilon)\, g^2 \mu^{2\varepsilon} C_F \delta_{\alpha\beta} \int \frac{d^d k}{(2\pi)^d} \frac{\not{p} + \not{k}}{(p+k)^2\, k^2}$$

Integrals

$$\Sigma(p) = 2(1-\varepsilon)\, g^2 \mu^{2\varepsilon} C_F \delta_{\alpha\beta} \int \frac{d^d k}{(2\pi)^d} \frac{\not{p} + \not{k}}{(p+k)^2\, k^2}$$
$$= 2(1-\varepsilon)\, g^2 \mu^{2\varepsilon} C_F \delta_{\alpha\beta} \left[\not{p} I + \gamma_\mu I^\mu \right] .$$

Define two integrals

$$\{I, I^\mu\} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(p+k)^2\, k^2} \{1, k^\mu\}$$

Feynman decomposition

We shall use Feynman trick $$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + (1-x)B]^2}$$

which gives:

$$\frac{1}{(p+k)^2\,k^2} = \int_0^1 dx \frac{1}{\left(k^2 + 2x\,p\cdot k + x\,p^2\right)^2}$$

$$= \int_0^1 dx \frac{1}{\left((k^2 + 2x\,p\cdot k + x^2p^2) + x(1-x)\,p^2\right)^2}$$

Shift integration variable $k^\mu \to k^\mu + xp^\mu$ and define $M^2 = -x(1-x)\,p^2$

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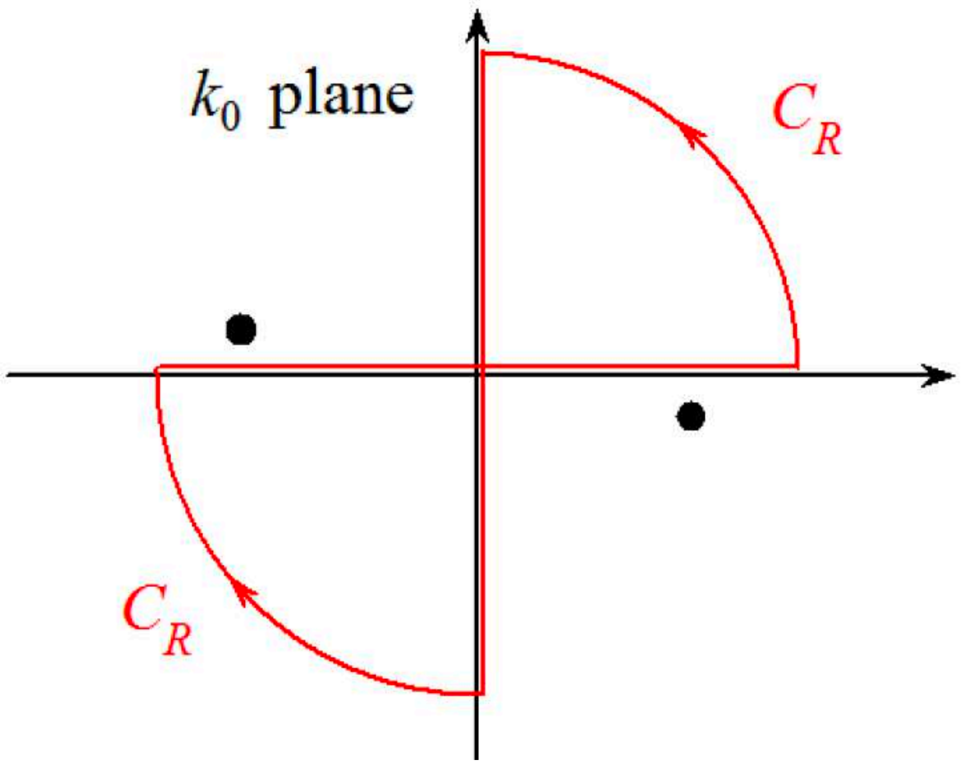
$$\{I, I^\mu\} = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left(k^2 - M^2\right)^2} \{1, k^\mu - xp^\mu\}$$

Wick rotation

We will change Minkowski integral to Euclidean

We have skipped Feynman *iε* prescription,
but now we have to recall where the poles are



$$\left\{ \int_{-\infty}^{\infty} + \int_{C_R} + \int_{+i\infty}^{-i\infty} \right\} dk^0 = 0$$

integral over C_R vanishes

$$\int_{-\infty}^{\infty} dk^0 = -\int_{+i\infty}^{-i\infty} dk^0 = i \int_{-\infty}^{+\infty} dE$$ where $k^0 = iE$ (integration limits!)

Wick rotation

Therefore Minkowski integrals

$$\{I, I^\mu\} = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - M^2)^2} \{1, k^\mu - xp^\mu\}$$

transform into *d* dimesional Euclidean integrals

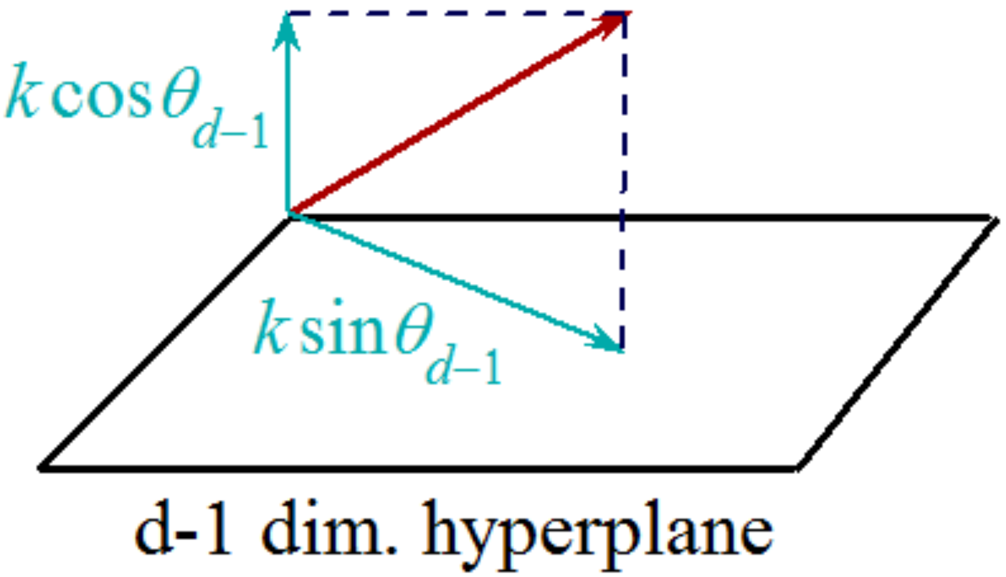
$$\{I, I^\mu\} = i \int_0^1 dx \int \frac{d^d \vec{k}}{(2\pi)^d} \frac{1}{\left(-\vec{k}^2 - M^2\right)^2} \{1, k^\mu - xp^\mu\}$$

where $\vec{k} = (E, k^1, k^2, \ldots, k^{d-1})$

Integrals in *d* dimensions

$$\begin{aligned} k_d &= k\cos\theta_{d-1}, \\ k_{d-1} &= k\sin\theta_{d-1}\cos\theta_{d-2}, \\ &\dots \\ k_2 &= k\sin\theta_{d-1}\sin\theta_{d-2}\dots\cos\theta_1 \\ k_1 &= k\sin\theta_{d-1}\sin\theta_{d-2}\dots\sin\theta_1 \end{aligned}$$

$$\theta_1 \in (0, 2\pi), \quad \theta_{i>1} \in (0, \pi)$$



Angular integration takes the following form

$$\int d\Omega_d = \int \prod_{i=1}^{d-1} (\sin^{i-1}\theta_i d\theta_i) = 2\prod_{i=1}^{d-1}\left(\int_0^\pi \sin^{i-1}\theta_i d\theta_i\right)$$

Usefull identities

$$\int_0^{\pi} \sin^n\theta\, d\theta = B\left(\frac{1+n}{2}, \frac{1}{2}\right)$$

Euler Beta function

$$\int_0^{\infty} dt\, \frac{t^{x-1}}{(1+t)^{x+y}} = B(x,y)$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\int_0^{1} dx\, x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta)$$

Usefull properties of Gamma functions

$$z\Gamma(z) = \Gamma(z+1),$$
$$\Gamma(1/2) = \sqrt{\pi}.$$

$$\Gamma(1-\varepsilon) = \exp\left(\gamma\varepsilon + \frac{\pi^2}{12}\varepsilon^2 + \ldots\right)$$

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$$\Gamma(\varepsilon) = \frac{1}{\varepsilon}\Gamma(1+\varepsilon)$$

Infinities show up as poles in ε, we have to remve poles before we go back to 4 dimensions