QCD

lecture 13

1 Heavy quark lagrangian

So far we have discussed effective QCD in the limit $m_{u,d,s} \to 0$ and $M_{c,b,t} \to \infty$ in a way that heavy quarks simply disapeared from the QCD Lagrangian. Here we would like to construct effective theory that will describe particles (mesoms or baryons) that contain at leas one heavy quark and some "light stuff". To this end we shall construct a pertinent lagragian, that will encode the relevant physics. It is clear that heavy quark inside a light hadron is almost on-shell and knows very little about the cloud of light quarks and gluons around. Therefore we shall parametrize heavy quark momentum in the following way:

$$p^\mu = M_Q v^\mu + k^\mu \tag{1}$$

where k^μ is a small (compared to M_Q) momentum that puts heavy quark off-shell. For on shell quark we have

$$p^2 = M_Q^2 v^2 + 2M_Q v \cdot k + k^2 = M_Q^2 \tag{2}$$

which implies

$$v^2 = 1,\ v \cdot k = 0,\ k^2 \ll M_Q^2. \tag{3}$$

This means that the "velocity" four-vector is normalized to 1. For off-shell quarks we allow $v \cdot k \neq 0$ but stll keep $k^2 \ll M_Q^2$. In this kinematics we can write heavy quark propagator in the following way:

$$\frac{i}{\not p - M_Q + i\varepsilon} = \frac{i\left(\not p + M_Q\right)}{p^2 - M_Q^2 + i\varepsilon} = \frac{1+\not v}{2}\frac{i}{v \cdot k + i\varepsilon} + \mathcal{O}\left(\frac{1}{M_Q}\right). \tag{4}$$

Note that the propagator in (4) does not contain M_Q and one can safely take the limt $M_Q \to \infty$. One can easily check that $(1+\not v)/2$ is a projection operator, because

$$\left(\frac{1+\not v}{2}\right)^2 = \frac{1}{4}\left(1 + 2\not v + \not v^2\right) = \frac{1+\not v}{2}. \tag{5}$$

This follows from the fact that $\not v^2 = v^2 = 1$. Similarly $(1-\not v)/2$ is also o projection operator and

$$\frac{1-\not v}{2}\frac{1+\not v}{2} = 0.$$

Before we answer what do these operators project out, let us consider a heavy quark

QCD vertex $-igT^a\gamma^\mu$. This vertex does not depend on M_Q but if it is inside a larger diagram it is sandwiched between the projectors from (4). A we will see external spinors will also have the projectors, so we need to calculate

$$\frac{1+\not{v}}{2}\gamma^\mu\frac{1+\not{v}}{2}$$

To this end we shall use

$$\not{v}\gamma^\mu = -\gamma^\mu\not{v} + 2v^\mu$$

and obtain

$$\begin{aligned} \frac{1+\not{v}}{2}\gamma^\mu\frac{1+\not{v}}{2} &= \gamma^\mu\frac{1-\not{v}}{2}\frac{1+\not{v}}{2} + v^\mu\frac{1+\not{v}}{2} \\ &= 0 + \frac{1+\not{v}}{2}v^\mu\frac{1+\not{v}}{2}. \end{aligned} \tag{6}$$

Therefore for heavy quarks

$$-igT^a\gamma^\mu \to -igT^a v^\mu. \tag{7}$$

Let us recall that in the Dirac representation for γ matrices solutions of Dirac equation take the following form

$$u(p,s) = \sqrt{E_p+m}\begin{bmatrix} \chi(s) \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m}\chi(s)\end{bmatrix}, \quad v(p,s) = \sqrt{E_p+m}\begin{bmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m}\chi(s) \\ \chi(s)\end{bmatrix}, \tag{8}$$

for quarks (bispinor u) and antiquarks (bispinor v). Here $\chi(s)$ are two component spinors corresponding to two projections of s_3 labelled by $s = \pm$. For a heavy quark at rest we have

$$u_Q(p,s) = \sqrt{2M_Q}\begin{bmatrix} \chi(s) \\ 0\end{bmatrix}, \quad v_Q(p,s) = \sqrt{2M_Q}\begin{bmatrix} 0 \\ \chi(s)\end{bmatrix} \tag{9}$$

and the projection operators read (recall that $v^\mu = (1,0,0,0)$ for heavy quark at rest)

$$\frac{1+\not{v}}{2} = \begin{pmatrix} \mathbf{1}_{2\times 2} & 0 \\ 0 & 0\end{pmatrix}, \quad \frac{1-\not{v}}{2} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_{2\times 2}\end{pmatrix}. \tag{10}$$

So wee see that the following is true in the heavy quark rest frane:

$$\frac{1+\not{v}}{2}u_Q(p,s) = u_Q(p,s), \ \frac{1-\not{v}}{2}v_Q(p,s) = v_Q(p,s). \tag{11}$$

Now, we extend this definition for any vector v. In the limit $m = M_Q \to \infty$

$$\frac{1+\not{v}}{2} u_Q(p,s) = u_Q(p,s) + \mathcal{O}\left(\frac{1}{m_Q}\right), \; \frac{1-\not{v}}{2} u_Q(p,s) = 0 + \mathcal{O}\left(\frac{1}{m_Q}\right). \tag{12}$$

Now we decompose the heavy quark field that enters the lagrangian for a heavy quark

$$\mathcal{L}_Q = \bar{Q}(x) \left(i\not{D} - M_Q\right) Q(x) \tag{13}$$

(where $D_\mu = \partial_\mu - ig\boldsymbol{A}_\mu$ is covariant QCD derivative) in the following way

$$Q(x) = e^{-iM_Q v\cdot x} \left[Q_v(x) + B_v(x)\right]$$

where

$$\begin{aligned} \frac{1+\not{v}}{2} Q_v(x) &= Q_v(x), \; \frac{1-\not{v}}{2} B_v(x) = B_v(x), \\ \frac{1-\not{v}}{2} Q_v(x) &= 0, \; \frac{1+\not{v}}{2} B_v(x) = 0. \end{aligned} \tag{14}$$

The second line of course follows from the first line due to the properties of the projection operators. Alternatively the same equations can written as

$$\not{v} Q_v(x) = Q_v(x), \; \not{v} B_v(x) = -B_v(x). \tag{15}$$

It is convenient to decompose the covariant derivative into two parts

$$i\not{D} = \not{v}\, iv\cdot D + i\not{D}_T. \tag{16}$$

Equation (16) can be in fact understood as a definition of D_T^μ

$$D_T^\mu = D^\mu - v^\mu v\cdot D, \tag{17}$$

which is indeed transverse to v: $v_\mu D_T^\mu = 0$.

We need to calculate now $i\not{D}Q(x)$. Obsrve that

$$i\partial_\mu e^{-iM_Q v\cdot x} \left[\ldots\right] = e^{-iM_Q v\cdot x} \left\{M_Q v_\mu + i\partial_\mu\right\} \left[\ldots\right].$$

We see now that

$$\begin{aligned}
iv\cdot\partial\, e^{-iM_Q v\cdot x}\,[\ldots] &= e^{-iM_Q v\cdot x}\,\{M_Q + i\,v\cdot\partial\}\,[\ldots] \\
i\partial^\mu_T\, e^{-iM_Q v\cdot x}\,[\ldots] &= e^{-iM_Q v\cdot x}\,\{M_Q v^\mu + i\partial^\mu - v^\mu M_Q - v^\mu i\,v\cdot\partial\}\,[\ldots] \\
&= e^{-iM_Q v\cdot x}\,\{i\partial^\mu - v^\mu i\,v\cdot\partial\}\,[\ldots]\,.
\end{aligned}$$

Hence

$$i\not{D} e^{-iM_Q v\cdot x}\,[\ldots] = e^{-iM_Q v\cdot x}\left\{M_Q\not{v} + \not{v}\,i\,v\cdot D + i\not{D}_T\right\}[\ldots]\,. \tag{18}$$

We could replace partial derivative by the covariant one because $e^{-iM_Q v\cdot x}$commutes with $-ig\not{\mathbf{A}}$. We can write the heavy quark lagrangian

$$\begin{aligned}
\mathcal{L}_Q &= \left[\bar{Q}_v(x) + \bar{B}_v(x)\right] e^{+iM_Q v\cdot x} e^{-iM_Q v\cdot x}\left\{\not{v}\,i\,v\cdot D + M_Q\not{v} + i\not{D}_T - M_Q\right\}\left[Q_v(x) + B_v(x)\right] \\
&= \left[\bar{Q}_v(x) + \bar{B}_v(x)\right]\left\{\not{v}\,i\,v\cdot D + M_Q\left(\not{v} - 1\right) + i\not{D}_T\right\}\left[Q_v(x) + B_v(x)\right]\,.
\end{aligned} \tag{19}$$

Now we shall apply to (19) relations (15)

$$\begin{aligned}
\mathcal{L}_Q &= \bar{Q}_v(x)\,i\,v\cdot D\,Q_v(x) - \bar{B}_v(x)\,\left(i\,v\cdot D + 2M_Q\right)\,B_v(x) \\
&\quad + \bar{Q}_v(x)\,i\not{D}_T B_v(x) + \bar{B}_v(x)\,i\not{D}_T Q_v(x).
\end{aligned} \tag{20}$$

Note that there is no mass term for Q_v.This follows from the fact that

$$M_Q\left(\not{v} - 1\right) Q_v = 0 \tag{21}$$

from (15). On the contrary

$$M_Q\left(\not{v} - 1\right) B_v(x) = -2M_Q B_v(x). \tag{22}$$

Note also that there are no diagonal terms of $i\not{D}_T$ (exercise). This follows from the identity (exercise)

$$\left(1 + \not{v}\right)\not{D}_T = \not{D}_T\left(1 - \not{v}\right), \tag{23}$$

Let's summarize what has been achieved so far:

- We have performed field redefinition at tree level. We correctly describe couplings to $k^\mu \sim \Lambda_{\rm QCD}$ gluons.
- Antiparticles are integrated out. They have $2M_Q$ mass gap.

- Possible mixing of B_v with *external* heavy quark Q_v is suppressed by $1/m_Q$.
- Number of heavy quarks is preserved (no $\mathrm{Q\bar{Q}}$ production).