QCD

lecture 12a

1 Goldberger-Treiman relation

Mass difference between neutron (939.56 MeV) and proton (938.27 MeV) is only 1.29 MeV. So small mass difference prohibits strong neutron decay, since the mass of the lightest negative (and positive) meson $\pi^{\mp}$ is 139.57 MeV. Therefore neutron can decay only weakly to electron and $\bar{\nu}_e$, since $m_e = 0.51$ MeV. Decay to muon is not possible because $m_\mu = 105.66$ MeV. Nevertheless it is clear that nucleon-pion coupling should be large, since strong coupling constant is approximately 10^6 larger that the weak coupling constant. Therefore, even though the W bosons couple directly to quarks (and therefore to nucleons), it maybe more profitable for a neutron to emit first a virtual pion, which then decays weakly. Such process is further enhanced due to the pion propagator that explodes for momenta of the order of the muon mass.

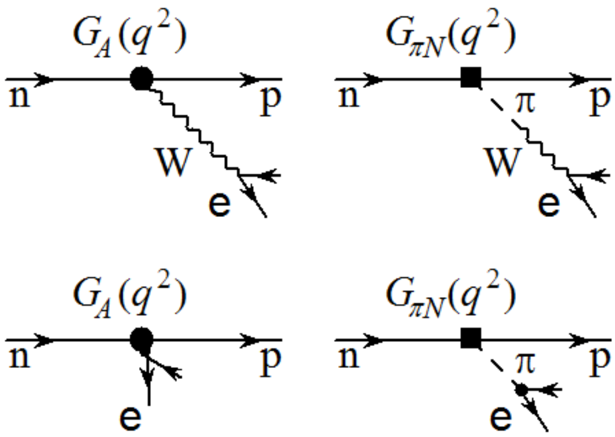

Figure 1: Neutron beta decay via direct coupling to W and via virtual pion decay.

Consider nucleon matrix element of the axial current, which can be parametrized in terms of two Dirac structures an therefore in terms of two effective couplings (from-factors)

$$\langle p(p')| A^a_\mu |n(p)\rangle = \bar{U}_n(p')\frac{\lambda_a}{2}\left[G_A(q^2)\gamma_\mu\gamma_5 + G_P(q^2)q_\mu\gamma_5\right]U_p(p) \tag{1}$$

and multiply it by $q = p - p'$. Note that spinors describing nucleons fulfill Dirac equation

$$\left(\not{p} - M\right)U_N(p) = 0 \tag{2}$$

where we assume that to a good approximation proton and neutron masses are equal to

M. We have therefore

$$q^{\mu} \langle p(p')| A^a_{\mu} |n(p)\rangle = \left(-2MG_A(q^2) + q^2 G_P(q^2)\right) \left(\bar{U}_n(p') \frac{\lambda_a}{2} \gamma_5 U_p(p)\right). \tag{3}$$

In the limit of unbroken chiral symmetry, which is our case here, since we assumed $M_n = M_p = M$ axial current is conserved, and we must have

$$-2MG_A(q^2) + q^2 G_P(q^2) = 0. \tag{4}$$

Consider the limit when $q^2 \to 0$. The first term is well measured experimentally: $g_A = G_A(0) = 1.257$. Although this numerical value is of course important, what is here of real imprtance is the fact that $G_A(0) \neq 0$, which implies that $G_P(q^2)$ must be of a form

$$G_P(q^2) = 2M \frac{g_A}{q^2} \tag{5}$$

for small q^2. The pole structure of the pseudoscalar form-factor $G_P(q^2)$ must be related to the (massless in this case) Goldstone boson (pion) propagator. Typically nucleon-pion interaction is introduced via the following vertex

$$g_{\pi NN} \left(\bar{U} i \tau_a \gamma_5 U\right)$$

and therefore the pion contribution to the axial current matrix element is equal to

$$i g_{\pi NN} \left(\bar{U} i \tau_a \gamma_5 U\right) \frac{i}{q^2} \left(i F_{\pi} q_{\mu}\right) \tag{6}$$

where the last term follows from the axial current matrix element between a pion state and vacuum. Factor i in front of the whole expression (6) follows from the convention concerning Feynman rules that renders the whole amplitude real. Comparing (6) with (5) and (3) we obtain *Goldberger-Treiman relation*

$$g_{\pi NN} F_{\pi} = M g_A \tag{7}$$

between nucleon axial coupling (in a sense direct coupling to W) and pion-nucleon coupling. Numerically $g^2_{\pi NN}/4\pi = 14.6$, and we obtain $g_{\pi NN} = 13.54$ and

$$\begin{aligned} g_{\pi NN} F_\pi &= M g_A \\ 1260 &= 1180 \end{aligned}$$

which means that GT relation is satisfied at the level of 7%.

2 Higher order lagrangians and loops

Effective lagrangian

$$\mathcal{L}_2 = \frac{F^2}{4} \operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) \tag{8}$$

describes GBs (pion in the SU(2) case) interactions that can be calculated by expanding

$$U = \exp\left(i \frac{\boldsymbol{\lambda} \cdot \boldsymbol{\phi}}{F}\right) \tag{9}$$

to any order in the number of fields ϕ. Coefficients of these terms are fixed by chiral symmetry (the lowest order term has to be canonical Klein-Gordon kinetic energy). Whatever the numer of fields there will be only two derivarives. For example the four-filed lagrangian (exercise) reads:

$$\mathcal{L}_2^{(4)} = \frac{1}{6F^2} \left\{ \left(\partial_\mu \boldsymbol{\phi} \cdot \boldsymbol{\phi}\right)\left(\partial^\mu \boldsymbol{\phi} \cdot \boldsymbol{\phi}\right) - \left(\partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi}\right)\left(\boldsymbol{\phi} \cdot \boldsymbol{\phi}\right) \right\}. \tag{10}$$

It is, however, possible to add to (8) terms with more derivatives. For example some of the four-derivative terms have the following form

$$\begin{aligned} \mathcal{L}_4 = & \; L_1 \left\{ \operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) \right\}^2 + L_2 \operatorname{Tr}\left(\partial_\mu U \partial_\nu U^\dagger\right) \operatorname{Tr}\left(\partial^\mu U \partial^\nu U^\dagger\right) \\ & + L_3 \operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger\right) + \ldots \end{aligned} \tag{11}$$

where dots stay for all other allowed by chiral symmetry and *inependent* terms. For an interacting theory ($\partial_\mu \to D_\mu$) there are alltogether 10 such terms (includng mass terms). Unlike in the case of (8) coefficients L_i are *not* constrained by chiral symmetry and have to be extracted from data.

There is, however, one problem with such effective theory: it is not renormalizable. This means that logarithmic (thanks God!) divergences cannot be removed by a finite number of renormalization constants. To see this, let's consider a loop correction to $\pi\pi$ scattering.

At first consider the lowest order interaction corresponding to (10) depicted in Fig. 2.

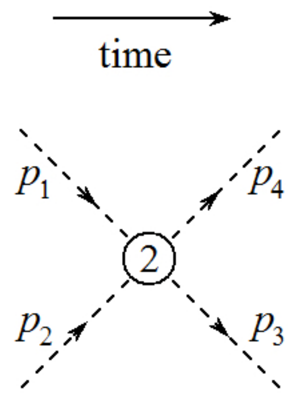

Figure 2: Pion-pion scattering. Arrows denote momentum flow. Label "2" inside the vertex blob indicates number of derivatives in the vertex.

The Feynman rule for this amplitude reads

$$\mathcal{M}\left(p_1, p_2, p_3, p_4\right) \sim \left(p_1+p_2\right) \cdot \left(p_3+p_4\right) - p_1 \cdot p_2 - p_3 \cdot p_4. \tag{12}$$

If we resca;e all momenta (and masses) $p \to tp$ this amplitude scales

$$\mathcal{M}\left(p_1, p_2, p_3, p_4\right) \to t^2 \mathcal{M}\left(p_1, p_2, p_3, p_4\right) \tag{13}$$

which reflects the fact that it corresponds two two derivative vertex.

Now consider a loop correction to $\pi\pi$ scattering depicted in Fig. 3. The corresponding amplitude (neglecting all irrelevant factors) takes the following form:

$$\begin{aligned} \mathcal{M}_{\text{loop}} \sim & \int d^4k \left[\left(p_1+p_2\right) \cdot \left(p_1+p_2\right) - p_1 \cdot p_2 - \left(p_1+p_2-k\right) \cdot k\right] \\ & \frac{1}{k^2-m^2} \frac{1}{\left(p_1+p_2-k\right)^2-m^2} \\ & \left[\left(p_1+p_2\right) \cdot \left(p_3+p_4\right) - \left(p_1+p_2-k\right) \cdot k - p_3 \cdot p_4\right] \end{aligned} \tag{14}$$

After rescaling

$$p_i \to tp_i,\ m \to tm,\ k \to tq$$

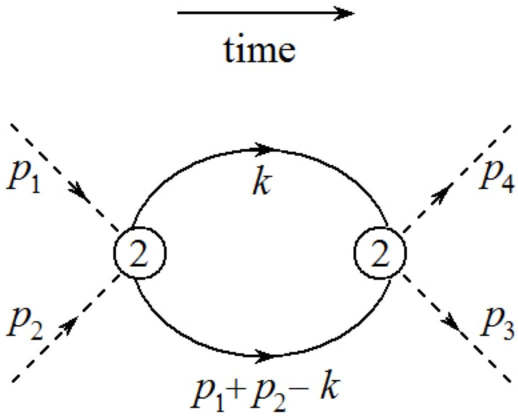

Figure 3: Loop contribution to pion-pion scattering. Arrows denote momentum flow. Label "2" inside the vertex blobs indicates number of derivatives in the vertex. Solid lines correspond to off-shell pions.

we obtain

$$\begin{aligned}\mathcal{M}_{\text{loop}} \quad \sim \quad & \overbrace{t^4}^{\text{vertices}}\ \overbrace{t^{-4}}^{\text{props}}\ \ \overbrace{t^4}^{\text{integration}} \\ & \int d^4l\ \left[(p_1+p_2)\cdot(p_1+p_2) - p_1\cdot p_2 - (p_1+p_2-l)\cdot l\right] \\ & \qquad \frac{1}{k^2-m^2}\frac{1}{(p_1+p_2-l)^2-m^2} \\ & \qquad \left[(p_1+p_2)\cdot(p_3+p_4) - (p_1+p_2-k)\cdot l - p_3\cdot p_4\right]. \end{aligned} \tag{15}$$

Hence

$$\mathcal{M}_{\text{loop}} \to t^4 \mathcal{M}_{\text{loop}}. \tag{16}$$

This means that the divergence enters at the level of four derivative terms. So it "renormalizes" coefficients L_i rather than the coefficeints following from $\mathcal{L}_2$. Hence we can absorb these divergences to unrenormalized (bare) constants $L_i \to L_i^r$. When we calculate loop corrections involving vertices from $\mathcal{L}_4$ (*i.e.* involving renormalized constants L_i^r) new divernces appear, but they affect some new couplings with higher number of derivatives, but not L_i^r's themselves. This theory is opperative at low energies, so we typcyally stop at the four derivative level (corresponding to p^4). This scheme is known as *chiral perturbation theory* and the resacling procedure introduced above is known as *Weinberg power counting.*

3 Linear sigma model

So far we have used an explicitly *non-linear* parametrization of the U matrix

$$U = U = \exp\left(i\frac{\boldsymbol{\lambda}\cdot\boldsymbol{\phi}}{F}\right).$$

It is possible, at least for SU(2) to use another paramtrization

$$U = \frac{1}{F}\left(\sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right) \tag{17}$$

with a constraint

$$\sigma^2 + \boldsymbol{\pi}^2 = F^2, \tag{18}$$

which follows from the unitarity of U. Expanding (8) in terms of (17) we will get different verces of $\pi\pi$ interactions. If external pions are on mass shell the amplitudes obtained from both parametrizations are the same (equivalence theorem). There exist methods for off-shell amplitudes (Gaser and Leutwyler: Green functions and Ward identities).