QCD

lecture 12

1 Adjoint representation: the story of two bases

We have already learned that group structure constants are group generators of the adjoint representation:

$$(T^a)_{bc} = -i f_{abc}. \tag{1}$$

Let's denote the basis corresponding to (1) $\{|\phi_a\rangle\}$ where $a = 1, 2, \dots N^2 - 1$(*natural* basis). We know, however, that in the case of SU(3) two out of eight generators (1) can be diagonalized, typically it is T^3 and T^8. To diagonalize these two generators we have to change the basis to the *physical* basis where states are numbered by hypercharge $Y = 2T^8/\sqrt{3}$, isospin $T(T+1) = \sum_{a=1}^{3} (T^a)^2$ and T^3. We have already encountered this problem in QM where we have two bases for spin 1 representation of SU(2), one defined in terms of epsilon symbols:

$$\langle\phi_C|\, \hat{J}_A \,|\phi_B\rangle = \left(\tilde{J}_A\right)_{CB} = -i\,\epsilon_{ACB}, \tag{2}$$

which explicitly read

$$\tilde{J}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \tilde{J}_2 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad \tilde{J}_3 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3}$$

and another one, corresponding to the basis where J_3 is diagonal:

$$\langle j, m'|\, \hat{J}_A \,|j, m\rangle = (J_A)_{m'm} \tag{4}$$

given as

$$J_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad J_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{5}$$

The relation between the two bases is given by a unitary transformation U. One can find U by diagonalizing $\tilde{J}_3$. Before we do this, let us see how matrices $\tilde{J}$ act on the states:

$$\hat{J}_A \,|\phi_B\rangle = \sum_D |\phi_D\rangle \left(\tilde{J}_A\right)_{DB} \tag{6}$$

with the sum running over D. Multiplying (6) by $\langle C|$ we obtain (2). We see therefore that, strictly speaking, basis states transform according to $\tilde{J}^T$ rather than $\tilde{J}$ and similarly for J. Note that – and this is what we are used to in QM – any state can be represented by a vector a_i constructed from the coefficients in the expansion

$$|\alpha\rangle = \sum_i |\phi_i\rangle\, a_i \;\rightarrow\; a_i = \langle\phi_i\, |\alpha\rangle$$

and then, for *any* operator $\hat{O}$ acting on $|\alpha\rangle$

$$|\beta\rangle = \hat{O}\,|\alpha\rangle$$

we have

$$b_k = \langle\phi_k|\, \beta\rangle = \sum_i \langle\phi_k|\, \hat{O}\,|\phi_i\rangle\, a_i = \sum_i O_{ki} a_i.$$

Let us define the unitary transformation between the two bases in the following way:

$$|1, m\rangle = \sum_B |\phi_B\rangle\, U_{Bm} \tag{7}$$

where $B = 1, 2, 3$ (summation) and m is fixed to be -1,0 or 1. So we have:

$$(J_A)_{m'm} = \langle 1, m'|\, \hat{J}_A\,|1, m\rangle = \sum_{C,B} U^*_{m'C}\,\langle\phi_C|\, \hat{J}_A\,|\phi_B\rangle\, U_{Bm} = \left(U^\dagger \tilde{J}_A U\right)_{m'm}. \tag{8}$$

The matrix U, which diagonalizes $\tilde{J}_3$ is not unique. One can choose freely the phases of the eigenvectors, and this leads to an ambiguity, that is fixed by some conventions. Here we just give a solution:

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{bmatrix}. \tag{9}$$

One can check by explicit calculation that matrices J_A are really reproduced. Plugging this form of U to Eq.(7) we obtain that:

$$\begin{aligned} |1\,,+1\rangle &= -\frac{1}{\sqrt{2}}\left(|\phi_1\rangle + i\ |\phi_2\rangle\right), \\ |1\,,0\rangle &= \qquad |\phi_3\ \rangle, \\ |1\,,-1\rangle &= +\frac{1}{\sqrt{2}}\left(|\phi_1\rangle - i\ |\phi_2\rangle\right). \end{aligned} \tag{10}$$

The minus sign in front of the first equation is important, and neglecting it may lead to errors in explicit calculations. The other signs are unequivocally determined by action of J_- operators. Of course in the case of isospin rather than spin eigenstates of J_3 correspond to pions:

$$\begin{aligned} \left|\pi^+\right\rangle &= |1\,,+1\rangle = -\frac{1}{\sqrt{2}}\left(|\phi_1\rangle + i\ |\phi_2\rangle\right) ,\\ \left|\pi^0\right\rangle &= |1\ ,0\rangle = \qquad |\phi_3\ \rangle ,\\ \left|\pi^-\right\rangle &= |1\,,-1\rangle = +\frac{1}{\sqrt{2}}\left(|\phi_1\rangle - i\ |\phi_2\rangle\right) . \end{aligned} \tag{11}$$

In SU(3) the situation is essentially the same, however, we have the 8-dimensional representation rather than a 3-dimensional one. The standard relation between the two bases is given by (De Alfaro, Fubini, Furlan, Rossetti *Currents in Hadron Physics*, 1973):

$$\begin{aligned} \left|K^+\right\rangle =\mid 8,\ \ 1,\frac{1}{2},+\frac{1}{2}\ \rangle &= -\frac{1}{\sqrt{2}}\left(\mid \phi_4\ \rangle + i\ \mid \phi_5\ \rangle\right) ,\\ \left|K^0\right\rangle =\mid 8,\ \ 1,\frac{1}{2},-\frac{1}{2}\ \rangle &= -\frac{1}{\sqrt{2}}\left(\mid \phi_6\ \rangle + i\ \mid \phi_7\ \rangle\right) ,\\ \left|\pi^+\right\rangle =\mid 8,\ \ 0,\ 1\,,+1\ \rangle &= -\frac{1}{\sqrt{2}}\left(\mid \phi_1\ \rangle + i\ \mid \phi_2\ \rangle\right) ,\\ \left|\pi^0\right\rangle =\mid 8,\ \ 0,\ 1\ ,\ \ 0\ \rangle &= \qquad \mid \phi_3\ \rangle\ ,\\ \left|\pi^-\right\rangle =\mid 8,\ \ 0,\ 1\,,-1\ \rangle &= +\frac{1}{\sqrt{2}}\left(\mid \phi_1\ \rangle - i\ \mid \phi_2\ \rangle\right) ,\\ \left|\bar{K}^0\right\rangle =\mid 8,\ -1,\frac{1}{2},+\frac{1}{2}\ \rangle &= -\frac{1}{\sqrt{2}}\left(\mid \phi_6\ \rangle - i\ \mid \phi_7\ \rangle\right) ,\\ \left|K^-\right\rangle =\mid 8,\ -1,\frac{1}{2},-\frac{1}{2}\ \rangle &= +\frac{1}{\sqrt{2}}\left(\mid \phi_4\ \rangle - i\ \mid \phi_5\ \rangle\right) ,\\ \left|\eta^0\right\rangle =\mid 8,\ \ 0,\ 0\ ,\ \ 0\ \rangle &= \qquad \mid \phi_8\ \rangle\ . \end{aligned} \tag{12}$$

So defined states fulfill De Swart's conjugation rule:

$$\mid (p,q),\ Y,I,I_3\ \rangle^* = (-)^Q \mid (q,p),\ -Y,I,-I_3\ \rangle\ ,$$

where $Q = I_3 + Y/2$.

With this convention matrix ϕ takes the following form in terms of physical states (we skip brackets for clarity):

$$
\begin{aligned}
\sum_a \phi_a \lambda_a &= \begin{pmatrix} \phi_3 + \phi_8/\sqrt{3} & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & -\phi_3 + \phi_8/\sqrt{3} & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -2\phi_8/\sqrt{3} \end{pmatrix} \\
&= \begin{pmatrix} \pi^0 + \eta^0/\sqrt{3} & \sqrt{2}\pi^- & \sqrt{2}K^- \\ -\sqrt{2}\pi^+ & -\pi^0 + \eta^0/\sqrt{3} & -\sqrt{2}\bar{K}^0 \\ -\sqrt{2}K^+ & -\sqrt{2}K^0 & -2\eta^0/\sqrt{3} \end{pmatrix}. \qquad (13)
\end{aligned}
$$

2 Gell-Mann, Oakes, Renner relation

Let's summarize results of the current algebra in the *natural* basis:

$$
\begin{aligned}
\partial^\mu A^a_\mu(x) &= i\bar{q}\left\{\frac{\lambda_a}{2}, M\right\}\gamma_5 q, \\
\langle 0|\, A^a_\mu(x)\, \left|\phi^b(p)\right\rangle &= ip_\mu e^{-ip\cdot x} F \delta^{ab}, \\
-F\left\langle \phi^b(p)\right|\, P_a(0)\, |0\rangle &= \frac{2}{3}\left\langle \bar{q}q\right\rangle \delta^{ab}, \\
P_a &= i\bar{q}\gamma_5\lambda_a q. \qquad (14)
\end{aligned}
$$

Now we shall translate these results to the *particle* basis for, say, π^-. For that we need

$$
\begin{aligned}
\frac{1}{2}\left(\lambda_1 + i\lambda_2\right) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv \lambda_+, \\
\phi_1 - i\phi_2 &= \sqrt{2}\pi^-. \qquad (15)
\end{aligned}
$$

We can now rewrite Eqs. (14) in terms of the physical state π^-. Let's start by rewriting the first equation(14):

$$
\begin{aligned}
A^+_\mu &= \bar{q}\gamma_\mu\gamma_5\lambda_+ q = \bar{q}\gamma_\mu\gamma_5\frac{\lambda_1 + i\lambda_2}{2} q = \bar{u}\gamma_\mu\gamma_5 d \\
i\bar{q}\left\{\lambda_+, M\right\}\gamma_5 q &= i\bar{q}\left\{\frac{\lambda_1 + i\lambda_2}{2}, M\right\}\gamma_5 q = i(m_u + m_d)\,\bar{u}\gamma_5 d
\end{aligned}
$$

because

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}+\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \quad (m_u+m_d)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\partial^\mu\left(\bar{u}\gamma_\mu\gamma_5 d\right) = i(m_u+m_d)\,\bar{u}\gamma_5 d. \tag{16}$$

Second equation (14)

$$\begin{aligned} \langle 0|\,(\bar{u}\gamma_\mu\gamma_5 d)(x)\left|\pi^-(p)\right\rangle &= \frac{1}{\sqrt{2}}\langle 0|\,A^1_\mu(x)+iA^2_\mu(x)\left|\phi^1(p)-i\phi^2(p)\right\rangle \\ &= i\frac{1}{\sqrt{2}}2\,p_\mu e^{-ip\cdot x}F = i\sqrt{2}\,p_\mu e^{-ip\cdot x}F. \end{aligned} \tag{17}$$

Differentiating second equation (14) and using (17) we get

$$\begin{aligned} \partial^\mu\left.\langle 0|\,\bar{u}\gamma_\mu\gamma_5 d\left|\pi^-(p)\right\rangle\right|_{x=0} &= \sqrt{2}\,p^2F = \sqrt{2}\,m_\pi^2F \\ \partial^\mu\left.\langle 0|\,\bar{u}\gamma_\mu\gamma_5 d\left|\pi^-(p)\right\rangle\right|_{x=0} &= (m_u+m_d)\underbrace{i\,\langle 0|\,\bar{u}\gamma_5 d\left|\pi^-(p)\right\rangle}_{\sqrt{2}G} \end{aligned} \tag{18}$$

and we get

$$m_\pi^2F = (m_u+m_d)G. \tag{19}$$

What is G? Recall (no summation over a):

$$\langle 0|\,i[Q^A_a(t),P_a]\,|0\rangle = -F\,\langle\phi^a|\,P_a\,|0\rangle = \frac{2}{3}\,\langle\bar{q}q\rangle\,.$$

Rewriting this in terms of the physical states we obtain

$$\begin{aligned} -F\,\langle 0|\underbrace{P^1+iP^2}_{2i\bar{u}\gamma_5 d}\underbrace{\left|\phi^1(p)-i\phi^2(p)\right\rangle}_{\sqrt{2}|\pi^-(p)\rangle} &= \frac{4}{3}\left\langle\bar{u}u+\bar{d}d+\bar{s}s\right\rangle = 4\,\langle\bar{u}u\rangle \\ -Fi\,\langle 0|\,\bar{u}\gamma_5 d\,\left|\pi^-(p)\right\rangle &= \sqrt{2}\,\langle\bar{u}u\rangle \\ -FG &= \langle\bar{u}u\rangle\,. \end{aligned}$$

So we finally obtain Gell-Mann–Oakes–Renner relation:

$$m_\pi^2 = (m_u + m_d)\left(-\frac{\langle \bar{u}u\rangle}{F^2}\right).$$

Gell-Mann, Oakes and Renner (Gell-Mann, M; Oakes, R J and Renner, B.; *Behavior of current divergences under* SU(3)×SU(3). Phys. Rev. 175 (1968) 2195)

3 Pion decay

3.1 External currents

In order to compute pion decay we have to introduce weak interaction to the effective action of QCD. First, we introduce into the QCD lagrangian couplings to *external* currents:

$$\mathcal{L} = \mathcal{L}^0_{\mathrm{QCD}} + \mathcal{L}_{\mathrm{ext}} = \mathcal{L}^0_{\mathrm{QCD}} + \bar{q}\left\{ \not{v} + \frac{1}{3}\not{v}_{(s)} + \gamma_5 \not{a} - s + i\gamma_5 p \right\} q \tag{20}$$

where

$$v^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} v_a^\mu, \quad a^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} a_a^\mu, \quad s = \sum_{a=0}^{8} \lambda_a s_a^\mu, \quad p = \sum_{a=0}^{8} \lambda_a p_a^\mu. \tag{21}$$

Note that ordinary QCD lagrangian is obtained by setting $v^\mu = v^\mu_{(s)} = a^\mu = p = 0$ and $s = \mathrm{diag}(m_u, m_d, m_s)$. Define generating functional

$$Z[v, v_{(s)}, a, s, p] = \langle 0|\, T \exp\left[i \int d^4x\, \mathcal{L}_{\mathrm{ext}}(x)\right] |0\rangle\,. \tag{22}$$

Differentiating Z with respect to the currents we can obtain Green functions. For example vacuum expectation value of the u quark condensate can be computed in the following way

$$\begin{aligned} &\langle 0|\, \bar{u}(x)u(x)\, |0\rangle_0 \\ = \;& \frac{i}{2}\left[\sqrt{\frac{2}{3}}\frac{\delta}{\delta s_0(x)} + \frac{\delta}{\delta s_3(x)} + \sqrt{\frac{1}{3}}\frac{\delta}{\delta s_8(x)}\right] Z[v, v_{(s)}, a, s, p]\Big|_{v=v_{(s)}=a=s=p=0}\,. \end{aligned} \tag{23}$$

It is convenient to rewrite $\mathcal{L}_{\mathrm{ext}}$ in terms of left and right currents defined as

$$v_\mu = \frac{1}{2}\left(r_\mu + l_\mu\right), \; a_\mu = \frac{1}{2}\left(r_\mu - l_\mu\right). \tag{24}$$

We have then

$$\mathcal{L}_{\rm ext} = \bar{q}_L \left(\not{l} + \frac{1}{3}\not{v}_s \right) q_L + \bar{q}_R \left(\not{r} + \frac{1}{3}\not{v}_s \right) q_R$$
$$-\bar{q}_R \left(s + ip \right) q_L - \bar{q}_L \left(s - ip \right) q_R. \tag{25}$$

In order to find conserved currents we observe that $\mathcal{L}^0_{\rm QCD} + \mathcal{L}_{\rm ext}$ is invariant under *local* transformations

$$q_R \to e^{-i\theta(x)/3} V_R(x) q_R,\ q_L \to e^{-i\theta(x)/3} V_L(x) q_L \tag{26}$$

where $V_{R,L}(x)$ are independent SU(3) matrices if

$$\begin{aligned} r_\mu &\to V_R r_\mu V_R^\dagger + i V_R \partial_\mu V_R^\dagger, \\ l_\mu &\to V_L l_\mu V_L^\dagger + i V_L \partial_\mu V_L^\dagger, \\ v_\mu^{(s)} &\to v_\mu^{(s)} - \partial_\mu \theta, \\ s + ip &\to V_R \left(s + ip \right) V_L^\dagger, \\ s - ip &\to V_L \left(s + ip \right) V_R^\dagger. \end{aligned} \tag{27}$$

3.2 Weak interactions of quarks

There is yet another practical use of local invariance, namely it allows to discuss couplings to external gauge fields in effective theory of QCD and to calculate *e.g.* weak pion decay: $\pi^- \to \mu^- \bar{\nu}_\mu$. Pion decay proceeds via coupling to the charged weak bosons

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \mp i W_\mu^2 \right). \tag{28}$$

Note that this is a different convention than in (13):

$$\boldsymbol{W}_\mu = \sum_a W_\mu^a \tau^a = \begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3 \end{pmatrix} = \begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & -W_\mu^3 \end{pmatrix}. \tag{29}$$

Recall, that in the case of weak interactions we have to distinguish couplings to left and right quarks. In this case the pertinent quark-W interaction reads

$$
\begin{aligned}
\mathcal{L}_{qW} &= -\frac{g}{\sqrt{2}} \begin{bmatrix} \bar{u}_L & \bar{c}_L & \bar{t}_L \end{bmatrix} \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \gamma^\mu \begin{bmatrix} d_L \\ s_L \\ b_L \end{bmatrix} W_\mu^+ + \text{h.c.} \\
&= -\frac{g}{\sqrt{2}} \left\{ \left[V_{ud} \left(\bar{u}_L \gamma^\mu d_L \right) + V_{us} \left(\bar{u}_L \gamma^\mu s_L \right) \right] W_\mu^+ + \left[V_{ud} \left(\bar{d}_L \gamma^\mu u_L \right) + V_{us} \left(\bar{s}_L \gamma^\mu u_L \right) \right] W_\mu^- \right\} + \ldots \\
&= -\frac{g}{\sqrt{2}} \begin{bmatrix} \bar{u}_L & \bar{d}_L & \bar{s}_L \end{bmatrix} \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \gamma^\mu \begin{bmatrix} u_L \\ d_L \\ s_L \end{bmatrix} W_\mu^+ + \text{h.c.}
\end{aligned}
$$

where $|V_{ud}| = 0.9735 \pm 0.0008, |V_{us}| = 0.2196 \pm 0.0023$ (Cabbibo angle). Defining

$$
T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; T_- = \begin{pmatrix} 0 & 0 & 0 \\ V_{ud} & 0 & 0 \\ V_{us} & 0 & 0 \end{pmatrix} \tag{30}
$$

the relevant lagragian can be written as

$$
\begin{aligned}
\mathcal{L}_{\text{int}} &= -\frac{g}{\sqrt{2}} \left(\bar{q}_L \gamma^\mu T_+ q_L \, W_\mu^+ + \bar{q}_L T_- q_L \, W^- \right) \\
&= -\frac{g}{\sqrt{2}} V_{ud} \left(\bar{u}_L \gamma^\mu d_L W_\mu^+ + \bar{d}_L \gamma^\mu u_L W_\mu^- \right) + \ldots
\end{aligned} \tag{31}
$$

From (31) we can read out the weak current

$$
l_\mu = -\frac{g}{\sqrt{2}} \left(T_+ W_\mu^+ + T_- W_\mu^- \right). \tag{32}
$$

Interaction vertex corresponding to the first term in (31) is shown in Fig. 1 below.

The other end of the W^- propagator in Fig. 1 has to be attached to the analogous lepton current describing W^- transition to $\mu^- + \bar{\nu}_\mu$.

Note that the interaction piece relevant for the π^- decay corresponds to first part of (31):

$$
V_{ud} \, \bar{u}_L \gamma^\mu d_L W_\mu^+ = \frac{1}{2} W_\mu^+ V_{ud} \, \bar{u} \gamma^\mu (1 - \gamma_5) d, \tag{33}
$$

which allows to define the pertinent current:

$$
J_\mu^{\pi^-} = \bar{u} \gamma^\mu (1 - \gamma_5) d
$$

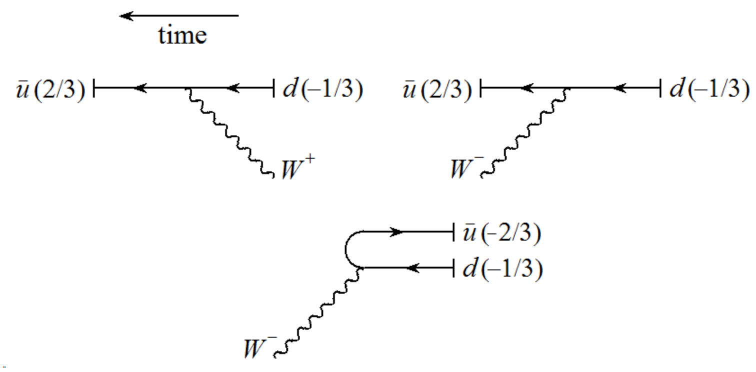

Figure 1: Upper left: illustration of the interaction vertex corresponding to the first term of Eq. (31). For an outgoing W boson one has to conjugate the W^+ field, which corresponds to W^- (upper right). Switching u quark to the initial state one obtains the vertex for π^- decay into W^- boson (bottom – note that in this case $\bar{u} = \bar{v}_u$).

and similarly for the muon-neutrino coupling

$$J_\tau^{\mu\bar{\nu}} = \bar{\nu}\gamma_\tau(1 - \gamma_5)\mu. \tag{34}$$

These two currents have to be contracted with the W propagator

$$\frac{-ig^{\mu\tau}}{k^2 - M_W^2} \simeq i\frac{g^{\mu\tau}}{M_W^2} \tag{35}$$

where in the last step we have approximated the propagator by the W mass, since $k^2 = m_\pi^2 \ll M_W^2$. Therefore the effective decay amplitude reads

$$\mathcal{M} = \frac{g^2}{8M_W^2} J_\tau^{\mu\bar{\nu}} J_{\pi^-}^\tau = \frac{1}{\sqrt{2}} G_F \, J_\tau^{\mu\bar{\nu}} J_{\pi^-}^\tau \tag{36}$$

where Fermi coupling constant is defined as

$$G_F = \frac{g^2}{8M_W^2}\sqrt{2} = 1.166 \times 10^{-5} \text{ GeV}^{-2}. \tag{37}$$

3.3 Weak interactions of Goldstone bosons

It may seem that the amplitude (36) can be used to calculate pion decay rate. Unfortunately it describes interaction of *free* quarks with the weak $\mu - \bar{\nu}$ current rather than the interaction of the Goldstne boson bound state. To circumvent this difficulty we shall now construct the weak pion current in the effective theory, where Goldstone fields are parametrized by a U matrix transforming as

$$U(x) \to V_R U(x) V_L^\dagger \tag{38}$$

where matrices $V_{L,R}$ are the same in QCD and in the effective theory. To this end we have to introduce couplings to the external currents l_μ and r_μ (and scalar densities as well), and these currents have to transform according to (27):

$$\begin{aligned} r_\mu &\to r'_\mu = V_R r_\mu V_R^\dagger + i V_R \partial_\mu V_R^\dagger, \\ l_\mu &\to l'_\mu = V_L l_\mu V_L^\dagger + i V_L \partial_\mu V_L^\dagger. \end{aligned}$$

Prescription is rather obvious:

$$\partial_\mu U \to D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu. \tag{39}$$

Let's check what is the transformation law of this covariant derivative, where – as previously – we promote the transformation to be local:

$$\begin{aligned} D_\mu U \;\to\; & \partial_\mu \left(V_R U V_L^\dagger\right) - i r'_\mu \left(V_R U V_L^\dagger\right) + i \left(V_R U V_L^\dagger\right) l'_\mu \\ = \; & \left(\partial_\mu V_R\right) U V_L^\dagger + V_R \left(\partial_\mu U\right) V_L^\dagger + V_R U \left(\partial_\mu V_L^\dagger\right) \\ & -i \left(V_R r_\mu V_R^\dagger + i V_R \partial_\mu V_R^\dagger\right) \left(V_R U V_L^\dagger\right) \\ & +i \left(V_R U V_L^\dagger\right) \left(V_L l_\mu V_L^\dagger + i V_L \partial_\mu V_L^\dagger\right) \end{aligned} \tag{40}$$

where in the last two lines we have used transformation laws of Eqs. (27). Let's rewrite the two last lines. For the third line in (40) we have to use

$$V_R \partial_\mu V_R^\dagger = - \left(\partial_\mu V_R\right) V_R^\dagger$$

to obtain

$$-i \left(V_R r_\mu V_R^\dagger - i \left(\partial_\mu V_R\right) V_R^\dagger\right) \left(V_R U V_L^\dagger\right) = -i V_R (r_\mu U) V_L^\dagger - \left(\partial_\mu V_R\right) U V_L^\dagger, \tag{41}$$

and next

$$+i\left(V_RUV_L^\dagger\right)\left(V_Ll_\mu V_L^\dagger+iV_L\partial_\mu V_L^\dagger\right)=+iV_R\left(Ul_\mu\right)V_L^\dagger-V_RU\left(\partial_\mu V_L^\dagger\right). \tag{42}$$

Comparing last two equations with (40) we see that terms involving derivatives $(\partial_\mu V_R)$ and $(\partial_\mu V_L^\dagger)$ cancel, and we are left with

$$\begin{aligned} D_\mu U &\to V_R\left(\partial_\mu U\right)V_L^\dagger-iV_R(r_\mu U)V_L^\dagger+iV_R\left(Ul_\mu\right)V_L^\dagger \\ &= V_R\left(\partial_\mu U-ir_\mu U+iUl_\mu\right)V_L^\dagger \\ &= V_R\left(D_\mu U\right)V_L^\dagger, \end{aligned} \tag{43}$$

which leaves

$$\mathrm{Tr}\left[\left(D_\mu U\right)\left(D^\mu U\right)^\dagger\right]$$

invariant.

Having introduced couplings to the external currents we can now caculate the weak interaction part of the effective lagrangian. We are interested only in the left (hermitean) current:

$$\begin{aligned} \mathcal{L} &= \frac{F_0^2}{4}\,\mathrm{Tr}\left[\left(D_\mu U\right)\left(D^\mu U\right)^\dagger\right] \\ &= \frac{F_0^2}{4}\,\mathrm{Tr}\left[\left(\partial_\mu U+iUl_\mu\right)\left(\partial^\mu U^\dagger-il^\mu U^\dagger\right)\right] \\ &= \frac{F_0^2}{4}\,\mathrm{Tr}\left[\left(\partial_\mu U\right)\left(\partial^\mu U^\dagger\right)\right]+i\frac{F_0^2}{4}\,\mathrm{Tr}\left[Ul_\mu\left(\partial^\mu U^\dagger\right)-\left(\partial_\mu U\right)l^\mu U^\dagger\right]. \end{aligned} \tag{44}$$

Since we are only interested in the interaction part, we can skip the first term. Using periodicity of trace and $U^\dagger\left(\partial_\mu U\right)=-\left(\partial^\mu U^\dagger\right)U$we arrive at

$$\mathcal{L}_{\mathrm{int}}=i\frac{F_0^2}{2}\,\mathrm{Tr}\left[l_\mu\left(\partial^\mu U^\dagger\right)U\right]. \tag{45}$$

Recall that we have already defined left current in effective theory:

$$J_L^\mu=i\frac{F_0^2}{4}\left(\partial^\mu U^\dagger\right)U, \tag{46}$$

which results in

$$\mathcal{L}_{\mathrm{int}}=2\,\mathrm{Tr}\left[l_\mu J_L^\mu\right]. \tag{47}$$

Decomposing, as in Eq. (21)

$$l_\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} l_\mu^a, \;\; J_L^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} J_L^{\mu,a} \tag{48}$$

we get

$$J_L^{\mu,a} = i\frac{F_0^2}{4}\,\mathrm{Tr}\left[\lambda_a\left(\partial^\mu U^\dagger\right)U\right] \tag{49}$$

finally arriving at

$$\mathcal{L}_{\rm int} = i\frac{F_0^2}{2} l_\mu^a\,\mathrm{Tr}\left[\frac{\lambda^a}{2}\left(\partial^\mu U^\dagger\right)U\right] = \sum_{a=1}^{8} l_\mu^a J_L^{\mu,a}. \tag{50}$$

Of course current $J_L^{\mu,a}$ is an infinite series in fields ϕ, we shall, however, keep only the leading term

$$J_L^{\mu,a} = \frac{F_0}{2}\partial^\mu\phi^a \text{ or } J_L^\mu = \frac{1}{4}F_0\partial^\mu\phi \tag{51}$$

We know from previous considerations the matrix element of axial current $J_A^{\mu,a} = -F_0\partial^\mu\phi^a$

$$\langle 0|\,J_A^{\mu,a}(x)\left|\phi^b(p)\right\rangle = ip^\mu e^{-ip\cdot x}F_0\delta^{ab} \tag{52}$$

but we need matrix element of the left current. Fortunately in the leading order we can use explicit form of $J_A^{\mu,a}$ to obtain:

$$\langle 0|\,\partial^\mu\phi^a\left|\phi^b(p)\right\rangle = -ip^\mu e^{-ip\cdot x}\delta^{ab} \tag{53}$$

which gives (for $x = 0$):

$$\langle 0|\,J_L^{\mu,a}\left|\phi^b(p)\right\rangle = -ip^\mu\frac{F_0}{2}\delta^{ab}. \tag{54}$$

Note that $|0\rangle$ is *strong* interaction vacuum, so it can contain W bosons and leptons.

In order to compute $W-\pi$ interaction we have to insert to (47) explicit form of l_μ weak current given by (32) and J_L^μ from Eq.(51). This gives:

$$\mathcal{L}_{\rm int} = -\frac{g}{\sqrt{2}}\frac{F_0}{2}\,\mathrm{Tr}\left[\left(W_\mu^+T_+ + W_\mu^-T_-\right)\partial^\mu\phi\right]. \tag{55}$$

We have to take matrix element $\langle 0|\,\mathcal{L}_{\rm int}\,|\pi^-(p)\rangle$ to which only the first term of (55) conributes

$$\begin{aligned}
\mathrm{Tr}\left[T_+\partial^\mu\phi\right] &= \left[\begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\partial^\mu\begin{pmatrix} \pi^0+\eta^0/\sqrt{3} & \sqrt{2}\pi^- & \sqrt{2}K^- \\ -\sqrt{2}\pi^+ & -\pi^0+\eta^0/\sqrt{3} & -\sqrt{2}\bar{K}^0 \\ -\sqrt{2}K^+ & -\sqrt{2}K^0 & -2\eta^0/\sqrt{3} \end{pmatrix}\right] \\
&= -\sqrt{2}V_{ud}\partial^\mu\pi^+ - \sqrt{2}V_{us}\partial^\mu K^+. \qquad (56)
\end{aligned}$$

Inserting this to (55) we obtain

$$\mathcal{L}_{\mathrm{int}} = gV_{ud}\frac{F_0}{2}\partial^\mu\pi^+ + \dots \qquad (57)$$

Note that

$$\pi^\mp = \pm\frac{1}{\sqrt{2}}\left(\phi^1 \mp i\phi^2\right). \qquad (58)$$

Therefore

$$\langle 0|\,\partial^\mu\pi^+\left|\pi^-(p)\right\rangle = -\frac{1}{2}\left(\langle 0|\,\partial^\mu\phi^1\left|\phi^1(p)\right\rangle + \langle 0|\,\partial^\mu\phi^2\left|\phi^2(p)\right\rangle\right) = ip^\mu \qquad (59)$$

because $\langle 0|\,\phi^a(x)\left|\phi^b(p)\right\rangle = \delta^{ab}e^{-ip\cdot x}$. Hence

$$\langle 0|\,\mathcal{L}_{\mathrm{int}}\left|\pi^-(p)\right\rangle = i\frac{1}{2}V_{ud}gF_0p^\mu\,W_\mu^+. \qquad (60)$$

According to our previous argument, where we discussed the approximation of the W propagator, we can write now the decay amplitude

$$\begin{aligned}
\mathcal{M} &\sim \frac{g}{2\sqrt{2}}\left(\bar{v}_{\nu_\mu}\gamma^\tau(1-\gamma_5)u_\mu\right)\frac{g_{\tau\mu}}{M_W^2}\frac{g}{2}V_{ud}F_0p^\mu \\
&= \underbrace{\frac{g^2}{8M_W^2}\sqrt{2}}_{=G_F}\,V_{ud}F_0\left(\bar{v}_{\nu_\mu}\not{p}(1-\gamma_5)u_\mu\right) \qquad (61)
\end{aligned}$$

where we were careless as far as all i factors and total sign are concerned, as they will cancel anyway when we calculate $|\mathcal{M}|^2$. We now see that the troublesome matrix element of the quark operator in (36) has been replaced by $\sim F_0p^\mu$.

3.4 Pion decay width

In order to calculate pion decay rate we shall employ a textbook formula for the decay of a particle at rest. Denoting pion four-momentum $p = (m_\pi, 0, 0, 0)$ we have

$$\begin{aligned}
\Gamma &= \frac{1}{2m_\pi} \int |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(p - k_1 - k_2) \frac{d^4 k_1}{(2\pi)^4} (2\pi) \delta(k_1^2 - m_\mu^2) \frac{d^4 k_2}{(2\pi)^4} (2\pi) \delta(k_2^2 - m_\nu^2) \\
&= \frac{1}{32\pi^2 m_\pi} \int |\mathcal{M}|^2 \, d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2 \frac{\delta(m_\pi - \sqrt{\boldsymbol{k}_1^2 + m_\mu^2} - \sqrt{\boldsymbol{k}_2^2 + m_\nu^2})}{\sqrt{\boldsymbol{k}_1^2 + m_\mu^2}\sqrt{\boldsymbol{k}_2^2 + m_\nu^2}} \delta^{(3)}(\boldsymbol{k}_1 + \boldsymbol{k}_2) \\
&= \frac{1}{32\pi^2 m_\pi} \int |\mathcal{M}|^2 \, d^3\boldsymbol{k}_1 \frac{\delta(m_\pi - \sqrt{\boldsymbol{k}_1^2 + m_\mu^2} - \sqrt{\boldsymbol{k}_1^2 + m_\nu^2})}{\sqrt{\boldsymbol{k}_1^2 + m_\mu^2}\sqrt{\boldsymbol{k}_1^2 + m_\nu^2}}
\end{aligned} \tag{62}$$

where $\boldsymbol{k}_1$ is muon momentum, which in the following we shall denote as $\boldsymbol{k}$.We can now perform the integral in the spherical coordinates. As we shall see the integral over the angles can be performed as the amplitude $\mathcal{M}$ depends only on k^2 and pk (to be checked). Therefore

$$d^3\boldsymbol{k} = k^2 dk d\Omega = 4\pi k^2 dk$$

and we obtain

$$\Gamma = \frac{1}{8\pi m_\pi} \int |\mathcal{M}|^2 \, k^2 dk \frac{\delta(m_\pi - \sqrt{k^2 + m_\mu^2} - \sqrt{k^2 + m_\nu^2})}{\sqrt{k^2 + m_\mu^2}\sqrt{k^2 + m_\nu^2}}. \tag{63}$$

In order to make use of the last Dirac delta it is convenient to introduce the following variable:

$$u = \sqrt{k^2 + m_\mu^2} + \sqrt{k^2 + m_\nu^2}. \tag{64}$$

Jacobian of this change of variables is

$$\frac{du}{dk} = \frac{uk}{\sqrt{k^2 + m_\mu^2}\sqrt{k^2 + m_\nu^2}}, \tag{65}$$

hence

$$\begin{aligned}
\Gamma &= \frac{1}{8\pi m_\pi} \int |\mathcal{M}|^2 \frac{k}{u} du \, \delta(m_\pi - u) \\
&= \frac{1}{8\pi m_\pi^2} |\mathcal{M}|^2 \, k(u = m_\pi).
\end{aligned} \tag{66}$$

Here $k(u = m_\pi)$ is a solution of (64) for k. We can now profit from the fact that the neutrino mass is (nearly) zero, and rewrite (64) as

$$\begin{aligned}(m_\pi - k)^2 &= k^2 + m_\mu^2 \\ m_\pi^2 - 2m_\pi k &= m_\mu^2 \\ k &= \frac{1}{2m_\pi}\left(m_\pi^2 - m_\mu^2\right) \\ &= \frac{m_\pi}{2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right), \end{aligned} \tag{67}$$

which gives

$$\Gamma = \frac{1}{16\pi m_\pi}|\mathcal{M}|^2\left(1 - \frac{m_\mu^2}{m_\pi^2}\right). \tag{68}$$

Note that with k given by Eq.(67) muon energy is equal

$$E_k = \sqrt{k^2 + m_\mu^2} = \frac{m_\pi}{2}\left(1 + \frac{m_\mu^2}{m_\pi^2}\right). \tag{69}$$

The last step consists in calculation of $|\mathcal{M}|^2$. To this end we have to compute the amplitude conjugated to (61)

$$\begin{aligned}\left(\bar{v}_{\nu_\mu}\not{p}(1-\gamma_5)u_\mu\right)^\dagger &= \left(v_{\nu_\mu}^\dagger\gamma^0\not{p}(1-\gamma_5)u_\mu\right)^\dagger \\ &= \left(u_\mu^\dagger(1-\gamma_5)\gamma^0\not{p}\gamma^0\gamma^0 v_{\nu_\mu}\right) \\ &= \left(\bar{u}_\mu(1+\gamma_5)\not{p}v_{\nu_\mu}\right) \end{aligned} \tag{70}$$

where we have used identity

$$\not{p}^\dagger = \gamma^0\not{p}\gamma^0. \tag{71}$$

We get

$$|\mathcal{M}|^2 = G_F^2 V_{ud}^2 F_0^2 \left(\bar{u}_\mu(1+\gamma_5)\not{p}v_{\nu_\mu}\right)\left(\bar{v}_{\nu_\mu}\not{p}(1-\gamma_5)u_\mu\right). \tag{72}$$

This amplitude has to be summed over muon and neutrino polarizations. To this end we shall use (note that neutrino momentum is $k-p$):

$$\begin{aligned} \sum_{\text{pol}} v_{\nu_\mu}\bar{v}_{\nu_\mu} &= \not{k} - \not{p}, \\ \sum_{\text{pol}} u_\mu \bar{u}_\mu &= \not{k} + m_\mu. \end{aligned} \tag{73}$$

Therefore

$$\sum_{\text{pol}} |\mathcal{M}|^2 = G_F^2 V_{ud}^2 F_0^2 \ \mathrm{Tr}\left[(1+\gamma_5)\not{p}\left(\not{k}-\not{p}\right)\not{p}(1-\gamma_5)\left(\not{k}+m_\mu\right)\right]. \tag{74}$$

Trace can be calculated with the help of *e.g. Mathematica*

$$\begin{aligned} \mathrm{Tr}\,[\ldots] &= 8\left(p^2k^2 + p^2(kp) - 2(kp)^2\right) \\ &= 4m_\mu^2 m_\pi^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right). \end{aligned} \tag{75}$$

Plugging this result into (68) we obtain final result for the pion decay width

$$\begin{aligned} \Gamma &= \frac{1}{16\pi m_\pi} G_F^2 V_{ud}^2 F_0^2 \ 4m_\mu^2 m_\pi^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)\left(1 - \frac{m_\mu^2}{m_\pi^2}\right) \\ &= \frac{G_F^2 V_{ud}^2 F_0^2}{4\pi} m_\mu^2 m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2. \end{aligned} \tag{76}$$

Note that decay to electron is suppressed by factor $(m_e/m_\mu)^2 \sim (1/200)^2$. Why is that so, we explainin more detail below.

Charged pion life-time is experimentally equal to $\tau_{\pi^\pm} = 2.6 \times 10^{-8}$sec. which is two orders shorter than the muon life-time 2.2×10^{-6} sec. However neutral pion life-time, that decays mostly electromagnetically to two photons is 8.52×10^{-17} sec. We can translate these lifetimes to decay widths by restoring $\hbar = 6.582 \times 10^{-22}$ MeV×sec. This gives

$$\Gamma_{\pi^\pm} = \frac{\hbar}{\tau_{\pi^\pm}} = 2.53 \times 10^{-14}\,\mathrm{MeV}. \tag{77}$$

From this numerical value w get indeed that $F_0 \sim 93$ MeV.

3.5 Chirality and helicity

In this section we shall give deeper insight into the question, why pion decay to electron, which is lighter than muon, is suppressed by factor $(m_e/m_\mu)^2 \sim (1/200)^2$ .To this end let's consider Dirac equation in the chiral representation of gamma matrices:

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}, \; \gamma_5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{78}$$

In this representation Dirac equation splits into two equations

$$(i\partial_t - i\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\, \psi_L - m\psi_R = 0, \qquad (i\partial_t + i\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\, \psi_R - m\psi_L = 0, \tag{79}$$

where the four-dimensional Dirac bispinors have been decomposed into two-dimensional Weyl spinors:

$$\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}. \tag{80}$$

For massless fermions, $m = 0$, (79) split into *independent* equations for $\psi_{L,R}$ separately, which reflects chrial structure of QCD. Recall that by introducing projection operators

$$P_L = \frac{1}{2}(1 - \gamma_5), \; P_R = \frac{1}{2}(1 + \gamma_5). \tag{81}$$

we can define chiral bispinors

$$\psi_- = \begin{bmatrix} \psi_L \\ 0 \end{bmatrix}, \qquad \psi_+ = \begin{bmatrix} 0 \\ \psi_R \end{bmatrix}. \tag{82}$$

Using (78) we see that $\psi_\mp$ are eigenstates of γ_5 for eigenvalue ∓ 1 respectively. We call this eigenvalue *chirality*. Note that chirality can be defined also for massive fermions.

For massless fermions we can introduce another conserved quantum number, namely *helicity*, which is defined as appropriately normalized spin projection on particle's momentum:

$$h = \frac{2}{p}\boldsymbol{p} \cdot \boldsymbol{\Sigma} = \frac{1}{p} \begin{bmatrix} \boldsymbol{p} \cdot \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{p} \cdot \boldsymbol{\sigma} \end{bmatrix}, \tag{83}$$

where $p = |\boldsymbol{p}|$. Free Dirac equation for massless fermions can be rewritten as:

$$\left(\gamma^0 E - \boldsymbol{\gamma}\cdot\boldsymbol{p}\right)\psi_\pm = 0 \quad \rightarrow \quad \frac{\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p}}{E}\psi_\pm = \psi_\pm. \tag{84}$$

Using chiral representation for Dirac matrices (78) it is easy to show

$$\gamma_5 h = \pm\frac{\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p}}{E_\pm} = \frac{1}{p}\begin{bmatrix} -\boldsymbol{p}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{p}\cdot\boldsymbol{\sigma} \end{bmatrix} \tag{85}$$

where $E_\pm = \pm p$. Indeed

$$\begin{aligned} \gamma_5 h &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{p}\begin{bmatrix} \boldsymbol{p}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{p}\cdot\boldsymbol{\sigma} \end{bmatrix} = \frac{1}{p}\begin{bmatrix} -\boldsymbol{p}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{p}\cdot\boldsymbol{\sigma} \end{bmatrix}, \\ \frac{\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{p}}{E} &= \frac{1}{E}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \boldsymbol{p}\cdot\boldsymbol{\sigma} \\ -\boldsymbol{p}\cdot\boldsymbol{\sigma} & 0 \end{bmatrix} = \frac{1}{E}\begin{bmatrix} -\boldsymbol{p}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{p}\cdot\boldsymbol{\sigma} \end{bmatrix}. \end{aligned}$$

Therefore for positive energy solutions (particles) Dirac equation (84) can be written in the following way:

$$\gamma_5 h\psi_\pm = \psi_\pm \quad \rightarrow \quad h\psi_\pm = \pm\psi_\pm, \tag{86}$$

and for negative energy solutions (antiparticles) as

$$-\gamma_5 h\psi_\pm = \psi_\pm \quad \rightarrow \quad h\psi_\pm = \mp\psi_\pm. \tag{87}$$

Therefore in the massless case helicity is equal to chirality for particles and to minus chirality for antiparticles.

Obviously for massive particles helicity is not a good quantum number. Helicity in one reference frame can be different from helicity in a different frame, since we can always choose an inertial reference frame that moves with velocity greater than the one of a particle in an initial frame. Such transformation flips particle's momentum, but leaves the direction of spin unchanged.

In the case of pion decay both charged lepton and antineutrino are left-handed, *i.e.* have chirality minus (note that weak coupling is proportional to $1-\gamma_5$). If not only antineutrino but also a charged fermion were massless, the decay would not take place. In pion rest frame lepton and antineutrino have opposite momenta, but lepton helicity is negative and antineutrino helicity is positive, which means that total spin projection on the axis of motion is equal 1, whereas initial spin of decaying pion is zero. For the

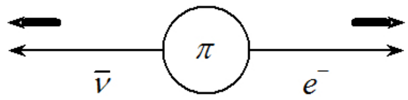

Figure 2: Pion decay. Antineutrino chirality is "$-$", but its helicity is "$+$". Because of the angular momentum conservation electron helicity should be also "$+$" (as depicted in the Figure), but this cannot happen for massless electron, whose chirality and helicity would be both "$-$". The decay may occur only because electron is not massless, and has a small helicity "$+$" component.

decay to take place lepton helicity (assuming still that antineutrino is massless) has to be positive. This is possible for massive particle and the amplitude for helicity flip is proportional, roughly speaking, to m_l/m_π. Hence probability of such process is proportional to $(m_l/m_\pi)^2$.