QCD lecture 11

December 22

Goldstone bosons

$$\langle 0|\, i[Q_a^A(t), P_a]\,|0\rangle \;=\; \frac{i}{2}\lim_{p^0\to 0}\sum_b \left\{ \frac{\langle 0|\, A_a^0\,|\phi^b\rangle}{p^0}\langle \phi^b|\, P_a\,|0\rangle - \langle 0|\, P_a\,|\phi^b\rangle \frac{\langle \phi^b|\, A_a^0\,|0\rangle}{p^0} \right\}$$

From hermicity and Lorentz invariance $\langle 0|\, A_a^\mu\,|\phi^b(p)\rangle = ip^\mu F_\phi \delta^{ab}$

and we get

$$\langle 0|\, i[Q_a^A(t), P_a]\,|0\rangle = -F_\phi \langle \phi^a|\, P_a\,|0\rangle = \frac{2}{3}\langle \bar{q}q\rangle$$

Here F_ϕ is Goldstone boson (pion) decay constant. Its value is ~ 93 MeV (different normalizations).

- There must exist states for which $\langle 0|\, A_a^0(0)\,|n\rangle$ and $\langle 0|\, P_a\,|n\rangle$ are non-zero
- It is not vacuum, because $\langle 0|\, P_a\,|0\rangle = 0$
- Energy of these states must vanish, because the quark condendate is time independent
- So we need $E_n = 0$
- Such states are massless Goldstone bosons $|\phi^b\rangle$
- GBs are (pseudo)scalars – still to be proven

Dimensions

Field dimensions:

$$\left[\int d^3x\mathcal{L}\right] = [\text{energy}] = 1 \qquad \left[d^3x\right] = \left[\text{distance}^3\right] = -3 \;\rightarrow\; [\mathcal{L}] = 4$$

$$\begin{aligned}4 &= [\mathcal{L}_D] = [\bar{q}\partial\!\!\!/ q] = [q]^2 + 1 \rightarrow \; [q] = \frac{3}{2} \;\rightarrow [\langle\bar{q}q\rangle] = 3\\ 4 &= [\mathcal{L}_{YM}] = [F_{\mu\nu}F^{\mu\nu}] = [F_{\mu\nu}]^2 \;\rightarrow\; [F_{\mu\nu}] = 2\\ 4 &= [\mathcal{L}_\phi] = \left[(\partial_\mu\phi)^2\right] \;\rightarrow\; [\phi] = 1\end{aligned}$$

Phenomenological values of condensates:

$$\begin{aligned}\langle\bar{q}q\rangle &\simeq -(250\,\mathrm{MeV})^3\\ \left\langle\frac{\alpha_s}{\pi}F^a_{\mu\nu}F^{a\,\mu\nu}\right\rangle &\simeq (\,400\,\mathrm{GeV})^4\end{aligned}$$

Dimension of currents

$$[J_\mu] = [\bar{q}\Gamma_\mu q] = 3$$

Dimensions

In the case of quantum fields there are different conventions. Here we follow:
T-P. Cheng and L-F. Li *Gauge theory of elementary particle physics*

$$\phi_a(\boldsymbol{x},t) \;=\; \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_k}} \left[a_a(\boldsymbol{k})e^{-ik\cdot x} + a_a^\dagger(\boldsymbol{k})e^{+ik\cdot x}\right]$$

$$[\phi_a] \;=\; 1 \rightarrow [a_a(\boldsymbol{k})] = -\frac{3}{2}$$

Indeed $\left[a_a(\boldsymbol{k}), a_a^\dagger(\boldsymbol{k}')\right] \;=\; \delta^{(3)}\left(\boldsymbol{k}-\boldsymbol{k}'\right)$

Fock state: $|\phi_a(k)\rangle \;=\; \sqrt{(2\pi)^3 2E_k}\, a_a^\dagger(\boldsymbol{k})\,|0\rangle \rightarrow [\,|\phi_a(k)\rangle] = -1$

Matrix element of axial current:

$$\left[\langle 0|\, J_A^{\mu,a}(0)\, \left|\phi^b(p)\right\rangle\right] \;=\; 3-1=2$$
$$\left[ip^\mu F_0 \delta^{ab}\right] \;=\; 2$$

Goldstone bosons

We have shown that in QCD axial SU(3) symmetry is spontaneously broken, and this implies the existence of eight Goldstone bosons. What is the effective lagrangian? Natural choice for example:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a^\dagger \partial^\mu \phi_a - V(\phi_a^\dagger \phi_a) \quad \phi'_a = \left[e^{-i\theta_c T^c_{\text{adj}}}\right]_{ab}\phi_b = \phi_a - i\theta_c \left(T^c_{\text{adj}}\right)_{ab}\phi_b + \ldots$$

This lagrangian is invariant under $SU_V(3)$ but it is not clear how it transforms under $SU_A(3)$. We will show, that we can write a lagrangian which is much more "powerfull" (infinte series in powers on field derivatives) and takes explicitly into account $SU_A(3)$ breaking. For this we will need a bit of mathematics.

Consider a hamiltonian $\hat{H}$ (note a "hat"!) which is invariant under a compact Lie group G Moreover, the ground state is invariant only under a subgroup H . We have therefore $n = n_G - n_H$ Goldstone bosons ϕ_i, which are continous, real functions on Minkowski space M^4. Define vector space

$$M_1 \equiv \{\Phi : M^4 \to R^n | \phi_i : M^4 \to R \text{ continuous}\}$$

and find its elements.

based on: Stefan Scherer *Introduction to Chiral Perturbation Theory*, hep-ph/0210398v1

Goldstone bosons

$$M_1 \equiv \{\Phi : M^4 \to R^n | \phi_i : M^4 \to R \text{ continuous}\}$$

Define a mapping that associates with each pair $(g, \Phi) \in G \times M_1$

g – group element,

Φ – *n* component vector with elements ϕ_i

an element $\varphi(g, \Phi) \in M_1$ such that

$$\varphi(e, \Phi) = \Phi \ \forall \ \Phi \in M_1, \ e \text{ identity of } G,$$
$$\varphi(g_1, \varphi(g_2, \Phi)) = \varphi(g_1 g_2, \Phi) \ \forall \ g_1, g_2 \in G, \ \forall \, \Phi \in M_1$$

This is nothing but definition of an operation of G on M_1 . This mapping is not necessarily linear:

$$\varphi(g, \lambda\Phi) \neq \lambda\varphi(g, \Phi)$$

Vacuum ("origin" of M_1) $\Phi = 0$ We require that all elements of *G* $h \in H$ map the origin onto itself (little group of $\Phi = 0$)

Goldstone bosons

- H is not empty, bcause identity maps the origin onto itself
- If $\varphi(h_1,0) = \varphi(h_2,0) = 0$ then $\varphi(h_1 h_2,0) = \varphi(h_1,\varphi(h_2,0)) = \varphi(h_1,0) = 0$ which means that $h_1 h_2 \in H$
- Inverse element is also in H: $\varphi(h^{-1},0) = \varphi(h^{-1},\varphi(h,0)) = \varphi(h^{-1}h,0) = \varphi(e,0)$ which means that $h^{-1} \in H$

Define left coset $gH = \{gH | g \in G\}$ (*g* is fixed) We will establish a connection between the set of all left cosets G/H with the Goldstone boson fileds.

We will check now that all elements of a given coset map the origin onto the same vector in R^n

$$\varphi(gh,0) = \varphi(g,\varphi(h,0)) = \varphi(g,0) \ \forall \ g \in G \text{ and } h \in H$$

These vectors are different if *g* and *g'* are "different": $\varphi(g,0) \neq \varphi(g',0)$ if $g' \notin gH$
This means that mapping φ is injective with respect to the cosets.

Goldstone bosons

Proof proceeds by negation of the thesis. Assume $\varphi(g,0) = \varphi(g',0)$
Then

$$0 = \varphi(e,0)$$

Goldstone bosons

Proof proceeds by negation of the thesis. Assume $\varphi(g,0) = \varphi(g',0)$
Then

$$0 = \varphi(e,0) = \varphi(g^{-1}g,0)$$

Goldstone bosons

Proof proceeds by negation of the thesis. Assume $\varphi(g, 0) = \varphi(g', 0)$
Then

$$0 = \varphi(e, 0) = \varphi(g^{-1}g, 0) = \varphi(g^{-1}, \varphi(g, 0))$$

Goldstone bosons

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Then

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Goldstone bosons

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Goldstone bosons

Proof proceeds by negation of the thesis. Assume $\varphi(g,0)=\varphi(g',0)$
Then

$$0=\varphi(e,0)=\varphi(g^{-1}g,0)=\varphi(g^{-1},\varphi(g,0))=\varphi(g^{-1},\varphi(g',0))=\varphi(g^{-1}g',0)$$

However, this implies $g^{-1}g'\in H$ or $g'\in gH$, which contradicts our assumption.

We will now discuss transformations of Φ. To each Φ corresponds a coset $\tilde{g}H$

with $\tilde{g}$ fixed:

$$\Phi=\varphi(f,0)=\varphi(\tilde{g}h,0)$$

Consider transformation of Φ with $\varphi(g)$

$$\varphi(g,\Phi)=\varphi(g,\varphi(\tilde{g}h,0))=\varphi(g\tilde{g}h,0)=\varphi(f',0)=\Phi' \qquad f'\in g(\tilde{g}H)$$

To obtain transformed Φ' from Φ we need to multiply the left coset $\tilde{g}H$ represinting Φ by g to obtain a new left coset represnting Φ'.

Goldstone bosons in QCD

Symmetry group of QCD

$$G = \mathrm{SU}(N) \times \mathrm{SU}(N) = \{(L,R)|L \in \mathrm{SU}(N), R \in \mathrm{SU}(N)\}$$

and little group $H = \{(V,V)|V \in \mathrm{SU}(N)\}$ (which is isomorphic to SU(N))

Left coset $\tilde{g}H = \{(\tilde{L}V, \tilde{R}V)|V \in \mathrm{SU}(N)\}$ is uniquely characterized by $U = \tilde{R}\tilde{L}^\dagger$

Indeed:

$$(\tilde{L}V, \tilde{R}V) = (\tilde{L}V, \tilde{R}\tilde{L}^\dagger\tilde{L}V) = (1, \tilde{R}\tilde{L}^\dagger)\underbrace{(\tilde{L}V, \tilde{L}V)}_{\in H}, \quad \text{i.e.} \quad \tilde{g}H = (1, \tilde{R}\tilde{L}^\dagger)H$$

(because $\varphi(gh,0) = \varphi(g,\varphi(h,0)) = \varphi(g,0)\ \forall\ g \in G$ and $h \in H$)

Therefore matrix $U = \tilde{R}\tilde{L}^\dagger$ is isomorphic to Φ.

Goldstone bosons in QCD

Now, we will find transformation law for U . Recall $\Phi = \varphi(f,0) = \varphi(\tilde{g}h,0)$

and $\varphi(f',0) = \Phi'$ where $f' = g\tilde{g}h$ or $f' \in g(\tilde{g}H)$. This means, that transformation

o U under $g = (L,R) \in G$ is (recall $\tilde{g}H = (1,\tilde{R}\tilde{L}^\dagger)H$)

$$g\tilde{g}H = (L, R\tilde{R}\tilde{L}^\dagger)H = (1, R\tilde{R}\tilde{L}^\dagger L^\dagger)\underbrace{(L,L)H}_{=\ H} = (1, R(\tilde{R}\tilde{L}^\dagger)L^\dagger)H$$

Hence we have $U = \tilde{R}\tilde{L}^\dagger \mapsto U' = R(\tilde{R}\tilde{L}^\dagger)L^\dagger = RUL^\dagger$

where we have to reintroduce space-time dependence

$$U(x) \mapsto RU(x)L^\dagger$$

We now see, how the symmetry is broken. Vacuum corresponds to $U \sim 1$ and the symmetry of vacuum is $R = L$.

Nonlinear realization of SU(*N*) x SU(*N*)

We can parametrize SU(*N*) matrix as $U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)$

where for SU(2)

$$\phi(x) = \sum_{i=1}^{3} \tau_i \phi_i(x) = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} \equiv \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$$

or for SU(3)

$$\phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) = \begin{pmatrix} \phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_1 - i\phi_2 & \phi_4 - i\phi_5 \\ \phi_1 + i\phi_2 & -\phi_3 + \frac{1}{\sqrt{3}}\phi_8 & \phi_6 - i\phi_7 \\ \phi_4 + i\phi_5 & \phi_6 + i\phi_7 & -\frac{2}{\sqrt{3}}\phi_8 \end{pmatrix}$$

$$\equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix},$$

[there exist different conventions for signs of particle fields]

Nonlinear realization of SU(*N*) x SU(*N*)

Define $M_3 \equiv \left\{U : M^4 \to \mathrm{SU}(N) | U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)\right\}$

The homomorphism

$$\varphi : G \times M_3 \to M_3 \quad \text{with} \quad \varphi[(L,R),U](x) \equiv RU(x)L^\dagger$$

defines an operation of G on M_3

1. $RUL^\dagger \in M_3$, since $U \in M_3$ and $R, L^\dagger \in \mathrm{SU}(N)$.
2. $\varphi[(1_{N\times N}, 1_{N\times N}), U](x) = 1_{N\times N}U(x)1_{N\times N} = U(x)$.
3. Let $g_i = (L_i, R_i) \in G$ and thus $g_1 g_2 = (L_1 L_2, R_1 R_2) \in G$.

$$\begin{aligned} \varphi[g_1, \varphi[g_2, U]](x) &= \varphi[g_1, (R_2 U L_2^\dagger)](x) = R_1 R_2 U(x) L_2^\dagger L_1^\dagger, \\ \varphi[g_1 g_2, U](x) &= R_1 R_2 U(x)(L_1 L_2)^\dagger = R_1 R_2 U(x) L_2^\dagger L_1^\dagger. \end{aligned}$$

all group requirements are fulfilled. This mapping is called nonlinear because M_3 is not a vector space (sum of two *U* matrices is not a unitary matrix).

Nonlinear realization of SU(*N*) x SU(*N*)

The origin (vacuum) corresponds to $\phi(x) = 0$, i.e. $U_0 = 1$

Indeed

$$\begin{aligned} \varphi[g = (V, V), 1] &= VV^\dagger = 1 \\ \varphi[g = (A, A^\dagger), 1] &= A^\dagger A^\dagger \neq 1 \end{aligned}$$

Axial symmetry is broken, left and right fermions must be transformed the same way.

Transformation of fieds $\phi(x)$

$$U = 1 + i\frac{\phi}{F_0} - \frac{\phi^2}{2F_0^2} + \cdots$$

and transformation matrix $V = \exp\left(-i\Theta_a^V \frac{\lambda_a}{2}\right)$ give

$$\phi = \lambda_b \phi_b \overset{h \in \mathrm{SU}(3)_V}{\mapsto} V\phi V^\dagger = \phi - i\Theta_a^V \underbrace{[\frac{\lambda_a}{2}, \phi_b \lambda_b]}_{\phi_b i f_{abc} \lambda_c} + \cdots = \phi + f_{abc}\Theta_a^V \phi_b \lambda_c + \cdots$$

Fields $\phi(x)$ transform according to the adjoint rep. of SU(3) (like gaue fields...)

Effective lagrangian

Matrix U is our "building block". Langrangian must be symmetric under global

$$\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V \qquad U(x) \mapsto RU(x)L^\dagger \quad U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)$$

The most general lagrangian with two derivatives (Weinberg lagrangian)

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4}\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)$$

where (experimentally) $F_0 \approx 93\ \mathrm{MeV}$ can be deduced from $\pi^+ \to \mu^+ \nu_\mu$

Invariance:

$$U \mapsto RUL^\dagger \quad \partial_\mu U \mapsto R\partial_\mu U L^\dagger \quad U^\dagger \mapsto LU^\dagger R^\dagger \quad \partial_\mu U^\dagger \mapsto L\partial_\mu U^\dagger R^\dagger$$

$$\mathcal{L}_{\text{eff}} \mapsto \frac{F_0^2}{4}\mathrm{Tr}\left(R\partial_\mu U \underbrace{L^\dagger L}_{1} \partial^\mu U^\dagger R^\dagger\right) = \frac{F_0^2}{4}\mathrm{Tr}\left(\underbrace{R^\dagger R}_{1} \partial_\mu U \partial^\mu U^\dagger\right) = \mathcal{L}_{\text{eff}}$$

Effective lagrangian

Expanding $U = 1 + i\phi/F_0 + \cdots \qquad \partial_\mu U = i\partial_\mu\phi/F_0 + \cdots$

$$\mathcal{L}_{\text{eff}} = \frac{F_0^2}{4}\text{Tr}\left[\frac{i\partial_\mu\phi}{F_0}\left(-\frac{i\partial^\mu\phi}{F_0}\right)\right] + \cdots = \frac{1}{4}\text{Tr}(\lambda_a\partial_\mu\phi_a\lambda_b\partial^\mu\phi_b) + \cdots$$

$$= \frac{1}{4}\partial_\mu\phi_a\partial^\mu\phi_b\text{Tr}(\lambda_a\lambda_b) + \cdots = \frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a + \mathcal{L}_{\text{int}}$$

we get usual lagrangian plus interactions that proceed only through derivatives (momenta). For small momenta higher derivative terms are small. Interactions are even in ϕ_a Parity

$$\phi_a(\vec{x},t) \mapsto -\phi_a(-\vec{x},t) \qquad U(\vec{x},t) \mapsto U^\dagger(-\vec{x},t)$$

This lagrangian is unique up to total derivatives. E.g.:

$$\text{Tr}[(\partial_\mu\partial^\mu U)U^\dagger] = \partial_\mu[\text{Tr}(\partial^\mu U U^\dagger)] - \text{Tr}(\partial^\mu U\partial_\mu U^\dagger)$$

Single derivatives vanish under trace $\text{Tr}(\partial_\mu U U^\dagger) = 0$

Currents

Left currents. Set $\Theta_a^R = 0$ and make left transformation space-time dependent: $\Theta_a^L = \Theta_a^L(x)$

Then
$$U \mapsto U' = RUL^\dagger = U\left(1 + i\Theta_a^L \frac{\lambda_a}{2}\right)$$

$$\partial_\mu U \mapsto \partial_\mu U' = \partial_\mu U\left(1 + i\Theta_a^L \frac{\lambda_a}{2}\right) + Ui\partial_\mu \Theta_a^L \frac{\lambda_a}{2}$$
$$\partial_\mu U^\dagger \mapsto \partial_\mu U'^\dagger = \left(1 - i\Theta_a^L \frac{\lambda_a}{2}\right)\partial_\mu U^\dagger - i\partial_\mu \Theta_a^L \frac{\lambda_a}{2} U^\dagger$$

and:
$$\begin{aligned}
\delta\mathcal{L}_{\text{eff}} &= \frac{F_0^2}{4}\text{Tr}\left[Ui\partial_\mu\Theta_a^L\frac{\lambda_a}{2}\partial^\mu U^\dagger + \partial_\mu U\left(-i\partial^\mu\Theta_a^L\frac{\lambda_a}{2}U^\dagger\right)\right] \\
&= \frac{F_0^2}{4}i\partial_\mu\Theta_a^L\text{Tr}\left[\frac{\lambda_a}{2}(\partial^\mu U^\dagger U - U^\dagger\partial^\mu U)\right] \\
&= \frac{F_0^2}{4}i\partial_\mu\Theta_a^L\text{Tr}\left(\lambda_a\partial^\mu U^\dagger U\right).
\end{aligned}$$

⟵ $\partial^\mu U^\dagger U = -U^\dagger \partial^\mu U$

Left current:
$$J_L^{\mu,a} = \frac{\partial\delta\mathcal{L}_{\text{eff}}}{\partial\partial_\mu\Theta_a^L} = i\frac{F_0^2}{4}\text{Tr}\left(\lambda_a\partial^\mu U^\dagger U\right)$$

Currents

Left currents. Set $\Theta_a^R = 0$ and make left transformation space-time dependent:

$\Theta_a^L = \Theta_a^L(x)$

Then
$$U \mapsto U' = RUL^\dagger = U\left(1 + i\Theta_a^L \frac{\lambda_a}{2}\right)$$

$$\partial_\mu U \mapsto \partial_\mu U' = \partial_\mu U\left(1 + i\Theta_a^L \frac{\lambda_a}{2}\right) + Ui\partial_\mu \Theta_a^L \frac{\lambda_a}{2}$$

$$\partial_\mu U^\dagger \mapsto \partial_\mu U'^\dagger = \left(1 - i\Theta_a^L \frac{\lambda_a}{2}\right)\partial_\mu U^\dagger - i\partial_\mu \Theta_a^L \frac{\lambda_a}{2} U^\dagger$$

and:
$$\begin{aligned}
\delta\mathcal{L}_{\text{eff}} &= \frac{F_0^2}{4}\text{Tr}\left[Ui\partial_\mu \Theta_a^L \frac{\lambda_a}{2}\partial^\mu U^\dagger + \partial_\mu U\left(-i\partial^\mu \Theta_a^L \frac{\lambda_a}{2} U^\dagger\right)\right] \\
&= \frac{F_0^2}{4} i\partial_\mu \Theta_a^L \text{Tr}\left[\frac{\lambda_a}{2}(\partial^\mu U^\dagger U - U^\dagger \partial^\mu U)\right] \\
&= \frac{F_0^2}{4} i\partial_\mu \Theta_a^L \text{Tr}\left(\lambda_a \partial^\mu U^\dagger U\right).
\end{aligned}$$

$\Leftarrow$ $\partial^\mu U^\dagger U = -U^\dagger \partial^\mu U$

Left current:
$$J_L^{\mu,a} = \frac{\partial \delta\mathcal{L}_{\text{eff}}}{\partial \partial_\mu \Theta_a^L} = i\frac{F_0^2}{4}\text{Tr}\left(\lambda_a \partial^\mu U^\dagger U\right)$$

Right current:
$$J_R^{\mu,a} = \frac{\partial \delta\mathcal{L}_{\text{eff}}}{\partial \partial_\mu \Theta_a^R} = -i\frac{F_0^2}{4}\text{Tr}\left(\lambda_a U \partial^\mu U^\dagger\right)$$

Currents

We can now calculate vector and axial currents:

$$
\begin{aligned}
J_V^{\mu,a} &= J_R^{\mu,a} + J_L^{\mu,a} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a[U,\partial^\mu U^\dagger]\right), \\
J_A^{\mu,a} &= J_R^{\mu,a} - J_L^{\mu,a} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a\{U,\partial^\mu U^\dagger\}\right)
\end{aligned}
$$

Internal parity:

$$
\begin{aligned}
J_V^{\mu,a} &\overset{\phi \mapsto -\phi}{\mapsto} -i\frac{F_0^2}{4}\mathrm{Tr}[\lambda_a(U^\dagger\partial^\mu U - \partial^\mu U U^\dagger)] \\
&= -i\frac{F_0^2}{4}\mathrm{Tr}[\lambda_a(-\partial^\mu U^\dagger U + U\partial^\mu U^\dagger)] = J_V^{\mu,a}
\end{aligned}
$$

$\partial^\mu U^\dagger U = -U^\dagger \partial^\mu U$ ⇒

$$
\begin{aligned}
J_A^{\mu,a} &\overset{\phi \mapsto -\phi}{\mapsto} -i\frac{F_0^2}{4}\mathrm{Tr}[\lambda_a(U^\dagger\partial^\mu U + \partial^\mu U U^\dagger)] \\
&= i\frac{F_0^2}{4}\mathrm{Tr}[\lambda_a(\partial^\mu U^\dagger U + U\partial^\mu U^\dagger)] = -J_A^{\mu,a}
\end{aligned}
$$

Matrix elemen of axial current

Axial current $J_A^{\mu,a} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a\{U, \partial^\mu U^\dagger\}\right)$

expanding: $J_A^{\mu,a} = -i\frac{F_0^2}{4}\mathrm{Tr}\left(\lambda_a\left\{1+\cdots, -i\frac{\lambda_b\partial^\mu\phi_b}{F_0}+\cdots\right\}\right) = -F_0\partial^\mu\phi_a+\cdots$

Matrix element of axial current between GB and vacuum:

$$\begin{aligned}\langle 0|\, J_A^{\mu,a}(x)\left|\phi^b(p)\right\rangle &= -F_0\,\langle 0|\,\partial^\mu\phi^a(x)\left|\phi^b(p)\right\rangle \\ &= -F_0\int\frac{d^4p'}{(2\pi)^4}\partial^\mu e^{-ip\cdot x}\underbrace{\langle 0|\,\phi^a(p')\left|\phi^b(p)\right\rangle}_{=(2\pi)^4\delta^{(4)}(p'-p)\,\delta^{ab}} \\ &= ip^\mu e^{-ip\cdot x}F_0\delta^{ab}.\end{aligned}$$

This agrees with previous result from QCD

$$\langle 0|A_\mu^a(0)|\phi^b(p)\rangle = ip_\mu F_0\delta^{ab}$$

Mass term

In QCD

$$\mathcal{L}_M = -\bar{q}_R M q_L - \bar{q}_L M^\dagger q_R, \quad M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

This would be invariant if $M \mapsto RML^\dagger$

What is the effective lagrangian that respects this would be symmetry? To the lowest order in M

$$\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2} \text{Tr}(MU^\dagger + UM^\dagger)$$

where B_0 is a new parameter. This means that the ground state $(U = 1)$ energy density is

$$\langle \mathcal{H}_{\text{eff}} \rangle = -F_0^2 B_0 (m_u + m_d + m_s)$$

In QCD

$$\left. \frac{\partial \langle 0 | \mathcal{H}_{\text{QCD}} | 0 \rangle}{\partial m_q} \right|_{m_u = m_d = m_s = 0} = \frac{1}{3} \langle 0 | \bar{q} q | 0 \rangle_0 = \frac{1}{3} \langle \bar{q} q \rangle$$

and we have

$$3 F_0^2 B_0 = -\langle \bar{q} q \rangle$$

Mass term

$$\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2} \text{Tr}(M U^\dagger + U M^\dagger) \qquad 3F_0^2 B_0 = -\langle \bar{q} q \rangle$$

Constant B_0 has dimension 1 (energy).

Expanding $U(x) \;=\; \exp\left(i\frac{\phi(x)}{F_0}\right)$ gives $\mathcal{L}_{\text{s.b}} = -\frac{B_0}{2}\text{Tr}(\phi^2 M) + \cdots$

Using

$$\phi(x) \;=\; \sum_{a=1}^{8} \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

one gets

$$\begin{aligned} \text{Tr}(\phi^2 M) \;=\; & 2(m_u + m_d)\pi^+\pi^- + 2(m_u + m_s)K^+K^- + 2(m_d + m_s)K^0\bar{K}^0 \\ & + (m_u + m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u - m_d)\pi^0\eta + \frac{m_u + m_d + 4m_s}{3}\eta^2. \end{aligned}$$

mixing

Mass term

$$\mathrm{Tr}(\phi^2 M) = 2(m_u+m_d)\pi^+\pi^- + 2(m_u+m_s)K^+K^- + 2(m_d+m_s)K^0\bar{K}^0 + (m_u+m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u-m_d)\pi^0\eta + \frac{m_u+m_d+4m_s}{3}\eta^2.$$

Isospin symmetric limit $m_u = m_d = m$

$\mathcal{L}_{\mathrm{s.b}} = -\frac{B_0}{2}\mathrm{Tr}(\phi^2 M)$ implies the following meson masses

$$M_\pi^2 = 2B_0 m,$$
$$M_K^2 = B_0(m+m_s),$$
$$M_\eta^2 = \frac{2}{3}B_0\,(m+2m_s)$$

where $B_0 = -\langle\bar{q}q\rangle/(3F_0^2)$

Gell-Mann – Okubo mass relation (does not depend on B_0)

$$4M_K^2 = 4B_0(m+m_s) = 2B_0(m+2m_s) + 2B_0 m = 3M_\eta^2 + M_\pi^2$$

$$L = 4\times 494^2 = 976\,144\,\mathrm{MeV}^2 \quad R = 3\times 548^2 + 138^2 = 919\,956\,\mathrm{MeV}^2$$

Mass term

$$\mathrm{Tr}(\phi^2 M) = 2(m_u+m_d)\pi^+\pi^- + 2(m_u+m_s)K^+K^- + 2(m_d+m_s)K^0\bar{K}^0$$
$$+(m_u+m_d)\pi^0\pi^0 + \frac{2}{\sqrt{3}}(m_u-m_d)\pi^0\eta + \frac{m_u+m_d+4m_s}{3}\eta^2.$$

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Gell-Mann – Okubo mass relation (does not depend on B_0)

$$\frac{L-R}{L+R} = 3\%$$

$$4M_K^2 = 4B_0(m+m_s) = 2B_0(m+2m_s) + 2B_0 m = 3M_\eta^2 + M_\pi^2$$

$L = 4\times 494^2 = 976\,144\,\mathrm{MeV}^2$ $R = 3\times 548^2 + 138^2 = 919\,956\,\mathrm{MeV}^2$

PCAC
partially conserved axial current

Let's calculate

$$\langle 0|\,\phi^a(x)\,|\phi^b(p)\rangle = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_k}} e^{-ik\cdot x}\,\langle 0|\,a_a(\boldsymbol{k})\sqrt{(2\pi)^3 2E_p}\,a_b^\dagger(\boldsymbol{p})\,|0\rangle$$
$$= \int d^3k \sqrt{\frac{E_p}{E_k}} e^{-ik\cdot x}\,\langle 0|\,a_a(\boldsymbol{k})a_b^\dagger(\boldsymbol{p})\,|0\rangle$$
$$= \delta^{ab} e^{-ip\cdot x}$$

Every field that has this property is called *interpolating field*.
Let's consider isospin subgroup of SU(3). Axial current matrix element

$$\langle 0|A_i^\mu(x)|\pi_j(q)\rangle = iq^\mu F_0 e^{-iq\cdot x}\delta_{ij}$$

Let's take its divergence

$$\langle 0|\partial_\mu A_i^\mu(x)|\pi_j(q)\rangle = iq^\mu F_0 \partial_\mu e^{-iq\cdot x}\delta_{ij} = M_\pi^2 F_0 e^{-iq\cdot x}\delta_{ij} = 2m_q B_0 F_0 e^{-iq\cdot x}\delta_{ij}$$

This means that divergence of the axial current, up to a constant, is itself pion interpolating field. On the other hand $\partial_\mu A_i^\mu = i\,m_q\,(\bar{q}\tau_i\gamma_5 q) = m_q P_i$
so pseudoscalar density is also a pion interpolating field.

PCAC
partially conserved axial current

Let's calculate

$$\langle 0|\,\phi^a(x)\,|\phi^b(p)\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}e^{-ik\cdot x}\,\langle 0|\,a_a(\boldsymbol{k})\sqrt{(2\pi)^3 2E_p}a_b^\dagger(\boldsymbol{p})$$

$$= \int d^3k\sqrt{\frac{E_p}{E_k}}e^{-ik\cdot x}\,\langle 0|\,a_a(\boldsymbol{k})a_b^\dagger(\boldsymbol{p})\,|0\rangle$$

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$$\langle 0|A_i^\mu(x)|\pi_j(q)\rangle = iq^\mu F_0 e^{-iq\cdot x}\delta_{ij}$$

$$-\frac{2}{3}\frac{m_q\,\langle\bar{q}q\rangle}{F_0}$$

Let's take its divergence

$$\langle 0|\partial_\mu A_i^\mu(x)|\pi_j(q)\rangle = iq^\mu F_0\partial_\mu e^{-iq\cdot x}\delta_{ij} = M_\pi^2 F_0 e^{-iq\cdot x}\delta_{ij} = 2m_q B_0 F_0 e^{-iq\cdot x}\delta_{ij}$$

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Chiral lagrangian

$$\mathcal{L} = \frac{F_0^2}{4}\,\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{F_0^2 B_0}{2}\,\mathrm{Tr}\left(M U^\dagger + U M\right) \qquad U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right)$$

Chiral lagrangian is expressed in terms of a U field

$$\phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}$$

- Nonzero quark condensate in chiral limit is a sufficient cond. for a spotaneus χSB
- Quark mass term gives masses to GBs
- Gell-Mann – Okubo mass formula emerges – satisfied experimentally
- Terms with more derivatives and with higher powers of M are possible
- Such theory is not renormalizable, but there is a method to make it predictive: chiral perturbation theory
- Coupling to photons, W and Z by covariant derivatives