QCD lecture 10

December 16

Conserved currents

We arrive at $\delta\mathcal{L} = \epsilon_a(x)\partial_\mu J^{\mu,a} + \partial_\mu\epsilon_a(x)J^{\mu,a}$

This allows do define currents and current derivarives as

$$
\begin{aligned}
J^{\mu,a} &= \frac{\partial\delta\mathcal{L}}{\partial\partial_\mu\epsilon_a},\\
\partial_\mu J^{\mu,a} &= \frac{\partial\delta\mathcal{L}}{\partial\epsilon_a}.
\end{aligned}
$$

If we demand the action to be invariant, we can integrate last term by parts, and we conclude that the current is conserved:

$$\partial_\mu J^{\mu,a} = 0$$

It follows that there exists a consrved charge (exercise)

$$Q^a(t) = \int d^3x J_0^a(\vec{x},t)$$

QCD currents

$$\begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} \mapsto U_L \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix} = \exp\left(-i\sum_{a=1}^{8}\Theta_a^L \frac{\lambda_a}{2}\right) e^{-i\Theta^L} \begin{pmatrix} u_L \\ d_L \\ s_L \end{pmatrix}$$

$$\begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} \mapsto U_R \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix} = \exp\left(-i\sum_{a=1}^{8}\Theta_a^R \frac{\lambda_a}{2}\right) e^{-i\Theta^R} \begin{pmatrix} u_R \\ d_R \\ s_R \end{pmatrix}$$

Repeating the same steps we arrive at (for massless fermions)

$$\delta\mathcal{L}^0_{\mathrm{QCD}} = \bar{q}_R \left(\sum_{a=1}^{8} \partial_\mu \Theta_a^R \frac{\lambda_a}{2} + \partial_\mu \Theta^R\right) \gamma^\mu q_R + \bar{q}_L \left(\sum_{a=1}^{8} \partial_\mu \Theta_a^L \frac{\lambda_a}{2} + \partial_\mu \Theta^L\right) \gamma^\mu q_L$$

and (quark fileds are now operators) we have 18 conserved currents:

$$\begin{aligned} L^{\mu,a} &= \bar{q}_L \gamma^\mu \frac{\lambda^a}{2} q_L, \quad \partial_\mu L^{\mu,a} = 0, \\ R^{\mu,a} &= \bar{q}_R \gamma^\mu \frac{\lambda^a}{2} q_R, \quad \partial_\mu R^{\mu,a} = 0. \end{aligned}$$

QCD currents

Define vector and axial currents

octet vector
$$V^{\mu,a} = R^{\mu,a} + L^{\mu,a} = \bar{q}\gamma^\mu \frac{\lambda^a}{2} q,$$

axial (exercise)
$$A^{\mu,a} = R^{\mu,a} - L^{\mu,a} = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda^a}{2} q,$$

singlet vector
$$V^\mu = \bar{q}_R \gamma^\mu q_R + \bar{q}_L \gamma^\mu q_L = \bar{q}\gamma^\mu q$$

axial (exercise)
$$A^\mu = \bar{q}_R \gamma^\mu q_R - \bar{q}_L \gamma^\mu q_L = \bar{q}\gamma^\mu \gamma_5 q$$

All these currents are conserved (modulo anomaly)

Parity of currents

Parity operator: γ^0

Transformation properties of gamma matrices

Γ	1	γ^μ	$\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$
$\gamma_0\Gamma\gamma_0$	1	γ_μ	$\sigma_{\mu\nu}$	$-\gamma_5$	$-\gamma_\mu\gamma_5$

imply the following properties of currents

$$P : V^{\mu,a}(\vec{x},t) \mapsto V^a_\mu(-\vec{x},t),$$
$$P : A^{\mu,a}(\vec{x},t) \mapsto -A^a_\mu(-\vec{x},t).$$

QCD charges

$$
\begin{aligned}
Q_L^a(t) &= \int d^3x\, q_L^\dagger(\vec{x},t)\frac{\lambda^a}{2}q_L(\vec{x},t), \quad a=1,\cdots,8, \\
Q_R^a(t) &= \int d^3x\, q_R^\dagger(\vec{x},t)\frac{\lambda^a}{2}q_R(\vec{x},t), \quad a=1,\cdots,8, \\
Q_V(t) &= \int d^3x \left[q_L^\dagger(\vec{x},t)q_L(\vec{x},t) + q_R^\dagger(\vec{x},t)q_R(\vec{x},t)\right].
\end{aligned}
$$

Recall anti-commutation relations for quark fields

$$
\begin{aligned}
\{q_{\alpha,r}(\vec{x},t), q_{\beta,s}^\dagger(\vec{y},t)\} &= \delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta}\delta_{rs} \\
\{q_{\alpha,r}(\vec{x},t), q_{\beta,s}(\vec{y},t)\} &= 0, \\
\{q_{\alpha,r}^\dagger(\vec{x},t), q_{\beta,s}^\dagger(\vec{y},t)\} &= 0,
\end{aligned}
$$

Commutators

To compute current commutators that are billinears in quark fields, we will use

$$\left[q^\dagger(\boldsymbol{x},t)\Gamma^{(1)}T^{(1)}q(\boldsymbol{x},t),q^\dagger(\boldsymbol{y},t)\Gamma^{(2)}T^{(2)}q(\boldsymbol{y},t)\right]$$
$$= \Gamma^{(1)}_{\alpha\beta}\Gamma^{(2)}_{\sigma\tau}\,T^{(1)}_{pq}T^{(2)}_{rs}\left[q^\dagger_{\alpha p}(\boldsymbol{x},t)q_{\beta q}(\boldsymbol{x},t),q^\dagger_{\sigma r}(\boldsymbol{y},t)q_{\tau s}(\boldsymbol{y},t)\right]$$

the identity

$$[ab,cd]=a\{b,c\}d-ac\{b,d\}+\{a,c\}db-c\{a,d\}b,$$

and cannonical anti-commutation rules

$$\begin{aligned}
\{q_{\alpha,r}(\vec{x},t),q^\dagger_{\beta,s}(\vec{y},t)\} &= \delta^3(\vec{x}-\vec{y})\delta_{\alpha\beta}\delta_{rs},\\
\{q_{\alpha,r}(\vec{x},t),q_{\beta,s}(\vec{y},t)\} &= 0,\\
\{q^\dagger_{\alpha,r}(\vec{x},t),q^\dagger_{\beta,s}(\vec{y},t)\} &= 0,
\end{aligned}$$

QCD commutation rules

QCD charges form Lie algebra (exercise)

$$\begin{aligned} [Q_L^a, Q_L^b] &= i f_{abc} Q_L^c, \\ [Q_R^a, Q_R^b] &= i f_{abc} Q_R^c, \\ [Q_L^a, Q_R^b] &= 0, \\ [Q_L^a, Q_V] &= [Q_R^a, Q_V] = 0 \end{aligned}$$

of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ group

For conserved charges:

$$[Q_L^a, H^0_{\mathrm{QCD}}] = [Q_R^a, H^0_{\mathrm{QCD}}] = [Q_V, H^0_{\mathrm{QCD}}] = 0$$

Axial current is anomalous, but otherwise it would commute with the hamiltonian as well.

QCD commutation rules

QCD charges form Lie algebra (exercise)

$$\begin{aligned} [Q_L^a, Q_L^b] &= i f_{abc} Q_L^c, \\ [Q_R^a, Q_R^b] &= i f_{abc} Q_R^c, \\ [Q_L^a, Q_R^b] &= 0, \\ [Q_L^a, Q_V] &= [Q_R^a, Q_V] = 0 \end{aligned}$$

$$\begin{aligned} [Q_V^a, Q_V^b] &= i f_{abc} Q_V^c \\ [Q_A^a, Q_A^b] &= i f_{abc} Q_V^c \\ [Q_V^a, Q_A^b] &= i f_{abc} Q_A^c \end{aligned}$$

of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ group

For conserved charges:

$$[Q_L^a, H_{\mathrm{QCD}}^0] = [Q_R^a, H_{\mathrm{QCD}}^0] = [Q_V, H_{\mathrm{QCD}}^0] = 0$$

Axial current is anomalous, but otherwise it would commute with the hamiltonian as well.

Quark masses – χSB

(chiral symmetry breaking)

$$M = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$
$$= \frac{m_u + m_d + m_s}{\sqrt{6}}\lambda_0 + \frac{(m_u + m_d)/2 - m_s}{\sqrt{3}}\lambda_8 + \frac{m_u - m_d}{2}\lambda_3.$$

$$\lambda_0 = \sqrt{\frac{2}{3}}\mathbf{1}$$

Symmetry breaking lagrangian:

$$\mathcal{L}_M = -\bar{q}Mq = -(\bar{q}_R M q_L + \bar{q}_L M q_R)$$

Now we calculate variation of $\mathcal{L}_M$ under chiral transformations

$$\exp\left(-i\sum_{a=1}^{8}\Theta_a^L\frac{\lambda_a}{2}\right)e^{-i\Theta^L}$$ and $$\exp\left(-i\sum_{a=1}^{8}\Theta_a^R\frac{\lambda_a}{2}\right)e^{-i\Theta^R}$$

Quark masses – χSB

(chiral symmetry breaking)

$$\delta\mathcal{L}_M = -i\left[\sum_{a=1}^{8}\Theta_a^R\left(\bar{q}_R\frac{\lambda_a}{2}Mq_L - \bar{q}_LM\frac{\lambda_a}{2}q_R\right) + \Theta^R\left(\bar{q}_RMq_L - \bar{q}_LMq_R\right)\right.$$
$$\left. + \sum_{a=1}^{8}\Theta_a^L\left(\bar{q}_L\frac{\lambda_a}{2}Mq_R - \bar{q}_RM\frac{\lambda_a}{2}q_L\right) + \Theta^L\left(\bar{q}_LMq_R - \bar{q}_RMq_L\right)\right],$$

From this we can easily calculate currents and current derivatives (lecture 9):

$$\partial_\mu L^{\mu,a} = \frac{\partial\delta\mathcal{L}_M}{\partial\Theta_a^L} = -i\left(\bar{q}_L\frac{\lambda_a}{2}Mq_R - \bar{q}_RM\frac{\lambda_a}{2}q_L\right),$$
$$\partial_\mu R^{\mu,a} = \frac{\partial\delta\mathcal{L}_M}{\partial\Theta_a^R} = -i\left(\bar{q}_R\frac{\lambda_a}{2}Mq_L - \bar{q}_LM\frac{\lambda_a}{2}q_R\right),$$
$$\partial_\mu L^{\mu} = \frac{\partial\delta\mathcal{L}_M}{\partial\Theta^L} = -i\left(\bar{q}_LMq_R - \bar{q}_RMq_L\right),$$
$$\partial_\mu R^{\mu} = \frac{\partial\delta\mathcal{L}_M}{\partial\Theta^R} = -i\left(\bar{q}_RMq_L - \bar{q}_LMq_R\right).$$

Quark masses – χSB

(chiral symmetry breaking)

$$
\begin{aligned}
\partial_\mu V^{\mu,a} &= i\bar{q}[M, \frac{\lambda_a}{2}]q, \\
\partial_\mu A^{\mu,a} &= i\left(\bar{q}_L\{\frac{\lambda_a}{2}, M\}q_R - \bar{q}_R\{\frac{\lambda_a}{2}, M\}q_L\right) = i\bar{q}\{\frac{\lambda_a}{2}, M\}\gamma_5 q, \\
\partial_\mu V^\mu &= 0, \\
\partial_\mu A^\mu &= 2i\bar{q}M\gamma_5 q + \text{ anomaly}
\end{aligned}
$$

Here we included anomaly, but for most of the time we will ignore it.

- Individual vector currents $\bar{u}\gamma^\mu u, \bar{d}\gamma^\mu d$ and $\bar{s}\gamma^\mu s$ are always conserved
- Vector current is a sum of them and is also conserved
- Baryon number is conserved
- Axial current is not conserved due to the quark masses (and anomaly)
- For equal quark mass all vector currents $V^{\mu,a}$ are conserved
- Axial flavor currents $A^{\mu,a}$ are not conserved, but their divergences are propotional to pseudoscalar densities. This leads to the concept of partially conserved axial currents (PCAC).

Chiral Ward identities

Define densities:

$$S_a(x) = \bar{q}(x)\lambda_a q(x), \quad P_a(x) = i\bar{q}(x)\gamma_5\lambda_a q(x), \quad a = 0, \cdots, 8$$

$$S(x) = \bar{q}(x)q(x), \quad P(x) = i\bar{q}(x)\gamma_5 q(x)$$

Ward identities relate divrgences of Green functions containing at least one current $V^{\mu,a}$ or $A^{\mu,a}$ to some linear combinations of other Green functions.

Example:

$$\begin{aligned} G_{AP}^{\mu,ab}(x,y) &= \langle 0|T[A_a^\mu(x)P_b(y)]|0\rangle \\ &= \Theta(x_0 - y_0)\langle 0|A_a^\mu(x)P_b(y)|0\rangle + \Theta(y_0 - x_0)\langle 0|P_b(y)A_a^\mu(x)|0\rangle \end{aligned}$$

We shall calculate: $\partial_\mu^x G_{AP}^{\mu,ab}(x,y)$ remembering that

$$\partial_\mu^x \Theta(x_0 - y_0) = \delta(x_0 - y_0)g_{0\mu} = -\partial_\mu^x \Theta(y_0 - x_0)$$

Chiral Ward identities

Differentiating

$$\begin{aligned} G^{\mu,ab}_{AP}(x,y) &= \langle 0|T[A^\mu_a(x)P_b(y)]|0\rangle \\ &= \Theta(x_0-y_0)\langle 0|A^\mu_a(x)P_b(y)|0\rangle + \Theta(y_0-x_0)\langle 0|P_b(y)A^\mu_a(x)|0\rangle \end{aligned}$$

we get:

$$\begin{aligned} &\partial^x_\mu G^{\mu,ab}_{AP}(x,y) \\ &= \delta(x_0-y_0)\langle 0|A^a_0(x)P_b(y)|0\rangle - \delta(x_0-y_0)\langle 0|P_b(y)A^a_0(x)|0\rangle \\ &\quad +\Theta(x_0-y_0)\langle 0|\partial^x_\mu A^\mu_a(x)P_b(y)|0\rangle + \Theta(y_0-x_0)\langle 0|P_b(y)\partial^x_\mu A^\mu_a(x)|0\rangle \\ &= \delta(x_0-y_0)\langle 0|[A^a_0(x),P_b(y)]|0\rangle + \langle 0|T[\partial^x_\mu A^\mu_a(x)P_b(y)]|0\rangle, \end{aligned}$$

equal time commutator can be calculated from chiral algebra.

time ordered product for conserved current this term is zero

Chiral Ward identities

Generalization

$$\begin{aligned}
\partial_\mu^x \langle 0|T\{J^\mu(x)A_1(x_1)\cdots A_n(x_n)\}|0\rangle = \\
= \langle 0|T\{[\partial_\mu^x J^\mu(x)]A_1(x_1)\cdots A_n(x_n)\}|0\rangle \\
+\delta(x^0 - x_1^0)\langle 0|T\{[J_0(x), A_1(x_1)]A_2(x_2)\cdots A_n(x_n)\}|0\rangle \\
+\delta(x^0 - x_2^0)\langle 0|T\{A_1(x_1)[J_0(x), A_2(x_2)]\cdots A_n(x_n)\}|0\rangle \\
+\cdots + \delta(x^0 - x_n^0)\langle 0|T\{A_1(x_1)\cdots [J_0(x), A_n(x_n)]\}|0\rangle
\end{aligned}$$

Current commutators

Full list:

$$
\begin{aligned}
[V_0^a(\vec{x},t), V_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} V_c^\mu(\vec{x},t), \\
[V_0^a(\vec{x},t), V^\mu(\vec{y},t)] &= 0, \\
[V_0^a(\vec{x},t), A_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} A_c^\mu(\vec{x},t), \\
[V_0^a(\vec{x},t), S_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} S_c(\vec{x},t), \\
[V_0^a(\vec{x},t), S_0(\vec{y},t)] &= 0, \\
[V_0^a(\vec{x},t), P_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} P_c(\vec{x},t), \\
[V_0^a(\vec{x},t), P_0(\vec{y},t)] &= 0, \\
[A_0^a(\vec{x},t), V_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} A_c^\mu(\vec{x},t), \\
[A_0^a(\vec{x},t), V^\mu(\vec{y},t)] &= 0, \\
[A_0^a(\vec{x},t), A_b^\mu(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} V_c^\mu(\vec{x},t), \\
[A_0^a(\vec{x},t), S_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} P_c(\vec{x},t), \\
[A_0^a(\vec{x},t), S_0(\vec{y},t)] &= 0, \\
[A_0^a(\vec{x},t), P_b(\vec{y},t)] &= \delta^3(\vec{x}-\vec{y}) i f_{abc} S_c(\vec{x},t), \\
[A_0^a(\vec{x},t), P_0(\vec{y},t)] &= 0.
\end{aligned}
$$

Schwinger terms*

Schwinger has shown that naive commutation rules involving charge densities have extra contributions:

$$[J_0^a(\vec{x},0), J_i^b(\vec{y},0)] = iC_{abc}J_i^c(\vec{x},0)\delta^3(\vec{x}-\vec{y}) + S_{ij}^{ab}(\vec{y},0)\partial^j\delta^3(\vec{x}-\vec{y}),$$

where the Schwinger term satisfies

$$S_{ij}^{ab}(\vec{y},0) = S_{ji}^{ba}(\vec{y},0)$$

One can get rid of the Schwinger terms by redefining the time ordered product. In what follows we shall ignore Schwinger terms.

S. Treiman, R. Jackiw, and D. J. Gross, *Lectures on Current Algebra and Its Applications* (Princeton University Press, Princeton, 1972).

Chiral Ward identities

Example: $$G^{\mu,ab}_{AP}(x,y) = \langle 0|T[A^\mu_a(x)P_b(y)]|0\rangle$$

we have shown: $$\partial^x_\mu G^{\mu,ab}_{AP}(x,y)$$
$$= \delta(x_0-y_0)\langle 0|[A^a_0(x),P_b(y)]|0\rangle + \langle 0|T[\partial^x_\mu A^\mu_a(x)P_b(y)]|0\rangle$$
$$= \delta^4(x-y)if_{abc}\langle 0|S_c(x)|0\rangle$$ < follows from symmetry

symmetry breaking > $$+i\langle 0|T[\bar{q}(x)\{\frac{\lambda_a}{2},M\}\gamma_5 q(x)P_b(y)]|0\rangle$$

We can now calculate the anti-commutator (no summation over a) [exercise]

$$i\bar{q}(x)\{\frac{\lambda_a}{2},M\}\gamma_5 q(x) =$$
$$\left[\frac{1}{3}(m_u+m_d+m_s)+\frac{1}{\sqrt{3}}\left(\frac{m_u+m_d}{2}-m_s\right)d_{aa8}\right]P_a(x)$$
$$+\left[\sqrt{\frac{1}{6}}(m_u-m_d)\delta_{a3}+\frac{\sqrt{2}}{3}\left(\frac{m_u+m_d}{2}-m_s\right)\delta_{a8}\right]P_0(x)$$
$$+\frac{m_u-m_d}{2}\sum_{c=1}^{8}d_{a3c}P_c(x).$$

Chiral Ward identities

Another example (for SU(2) and for $m_u = m_d = m$):

$$\partial^\mu A^i_\mu \;=\; im\left(\bar{q}\tau^i\gamma_5 q\right)$$

Consider nucleon matrix element

$$\langle N(p_f)|A^i_\mu(x)|N(p_i)\rangle = \langle N(p_f)|\bar{q}(x)\gamma_\mu\gamma_5\frac{\tau_i}{2}q(x)|N(p_i)\rangle$$

and take its derivatve

$$\begin{aligned}\partial^\mu\langle N(p_f)|\,A^i_\mu\,|N(p_i)\rangle \;&=\; im\,\langle N(p_f)|\,\bar{q}\tau^i\gamma_5 q\,|N(p_i)\rangle\\ &= m\,\langle N(p_f)|\,P_i\,|N(p_i)\rangle\end{aligned}$$

But nucleon matrix element of the pseudoscalar density can be related to the pion coupling to the nucleon (Goldberger-Treiman relation, to be discussed later)

QCD spectrum

Both vector and axial charges commute with QCD (massless) hamiltonian $H^0_{\rm QCD}$ therefore the eigenstates organize themselves into irreducible representations of the chiral group $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$ (axial $\mathrm{U}(1)$ is broken by anomaly). States within each multiplet are (nearly) degenarate in mass and have well defined baryon number ($\mathrm{U}(1)_V$ ensures baryon number conservation). Since axial and vector charges have opposite parity, one would expect that multiplets of opposite parity are degenerate in mass.

For positive parity states:
(e.g. baryon or meson ground sates)

$$H^0_{\rm QCD}|i,+\rangle = E_i|i,+\rangle$$

$$P|i,+\rangle = +|i,+\rangle$$

Define now* $|\phi\rangle = Q^a_A|i,+\rangle$ and calculate its mass. Because $[H^0_{\rm QCD}, Q^a_A] = 0$

$$H^0_{\rm QCD}|\phi\rangle = H^0_{\rm QCD}Q^a_A|i,+\rangle = Q^a_A H^0_{\rm QCD}|i,+\rangle = E_i Q^a_A|i,+\rangle = E_i|\phi\rangle$$

so the new state has the same energy (mass) but opposite parity

$$P|\phi\rangle = PQ^a_A P^{-1}P|i,+\rangle = -Q^a_A(+|i,+\rangle) = -|\phi\rangle$$

*charges and generators transforming Hilbert space states are identical (lecture 9)

QCD spectrum

State $|\phi\rangle$ can be expanded in the basis of negative parity multiplet (in fact generators are Clebsch-Gordan coefficients)

$$|\phi\rangle = Q^a_A |i, +\rangle = -t^a_{ij} |j, -\rangle$$

But such degeneracy of opposite parity states is not seen in Nature.

$n^{2s+1}\ell_J$	J^{PC}	I = 1 $u\bar{d}$, $\bar{u}d$, $\frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$	I = $\frac{1}{2}$ $u\bar{s}$, $d\bar{s}$; $\bar{d}s$, $\bar{u}s$	I = 0 f'	I = 0 f	θ_{quad} [°]	θ_{lin} [°]
1^1S_0	0^{-+}	$\pi(138)$	$K(494)$	$\eta(548)$	$\eta'(958)$	−11.3	−24.5
1^3S_1	1^{--}	$\rho(770)$	$K^*(892)$	$\phi(1020)$	$\omega(782)$	39.2	36.5
1^1P_1	1^{+-}	$b_1(1235)$	$K_{1B}{}^\dagger$	$h_1(1415)$	$h_1(1170)$		
1^3P_0	0^{++}	$a_0(1450)$	$K_0^*(1430)$	$f_0(1710)$	$f_0(1370)$		
1^3P_1	1^{++}	$a_1(1260)$	$K_{1A}{}^\dagger$	$f_1(1420)$	$f_1(1285)$		
1^3P_2	2^{++}	$a_2(1320)$	$K_2^*(1430)$	$f_2'(1525)$	$f_2(1270)$	29.6	28.0
1^1D_2	2^{-+}	$\pi_2(1670)$	$K_2(1770)^\dagger$	$\eta_2(1870)$	$\eta_2(1645)$		
1^3D_1	1^{--}	$\rho(1700)$	$K^*(1680)^\ddagger$		$\omega(1650)$		
1^3D_2	2^{--}		$K_2(1820)^\dagger$				
1^3D_3	3^{--}	$\rho_3(1690)$	$K_3^*(1780)$	$\phi_3(1850)$	$\omega_3(1670)$	31.8	30.8
1^3F_4	4^{++}	$a_4(1970)$	$K_4^*(2045)$	$f_4(2300)$	$f_4(2050)$		
1^3G_5	5^{--}	$\rho_5(2350)$	$K_5^*(2380)$				
2^1S_0	0^{-+}	$\pi(1300)$	$K(1460)$	$\eta(1475)$	$\eta(1295)$		
2^3S_1	1^{--}	$\rho(1450)$	$K^*(1410)^\ddagger$	$\phi(1680)$	$\omega(1420)$		
2^3P_1	1^{++}	$a_1(1640)$					
2^3P_2	2^{++}	$a_2(1700)$	$K_2^*(1980)$	$f_2(1950)$	$f_2(1640)$		

J^P	(D, L^P_N)	S	Octet members				Singlets
$1/2^+$	$(56,0^+_0)$	$1/2$	$N(939)$	$\Lambda(1116)$	$\Sigma(1193)$	$\Xi(1318)$	
$1/2^+$	$(56,0^+_2)$	$1/2$	$N(1440)$	$\Lambda(1600)$	$\Sigma(1660)$	$\Xi(1690)^\dagger$	
$1/2^-$	$(70,1^-_1)$	$1/2$	$N(1535)$	$\Lambda(1670)$	$\Sigma(1620)$	$\Xi(?)$	$\Lambda(1405)$
					$\Sigma(1560)^\dagger$		
$3/2^-$	$(70,1^-_1)$	$1/2$	$N(1520)$	$\Lambda(1690)$	$\Sigma(1670)$	$\Xi(1820)$	$\Lambda(1520)$
$1/2^-$	$(70,1^-_1)$	$3/2$	$N(1650)$	$\Lambda(1800)$	$\Sigma(1750)$	$\Xi(?)$	
					$\Sigma(1620)^\dagger$		
$3/2^-$	$(70,1^-_1)$	$3/2$	$N(1700)$	$\Lambda(?)$	$\Sigma(1940)^\dagger$	$\Xi(?)$	
$5/2^-$	$(70,1^-_1)$	$3/2$	$N(1675)$	$\Lambda(1830)$	$\Sigma(1775)$	$\Xi(1950)^\dagger$	
$1/2^+$	$(70,0^+_2)$	$1/2$	$N(1710)$	$\Lambda(1810)$	$\Sigma(1880)$	$\Xi(?)$	$\Lambda(1810)^\dagger$
$3/2^+$	$(56,2^+_2)$	$1/2$	$N(1720)$	$\Lambda(1890)$	$\Sigma(?)$	$\Xi(?)$	
$5/2^+$	$(56,2^+_2)$	$1/2$	$N(1680)$	$\Lambda(1820)$	$\Sigma(1915)$	$\Xi(2030)$	
$7/2^-$	$(70,3^-_3)$	$1/2$	$N(2190)$	$\Lambda(?)$	$\Sigma(?)$	$\Xi(?)$	$\Lambda(2100)$
$9/2^-$	$(70,3^-_3)$	$3/2$	$N(2250)$	$\Lambda(?)$	$\Sigma(?)$	$\Xi(?)$	
$9/2^+$	$(56,4^+_4)$	$1/2$	$N(2220)$	$\Lambda(2350)$	$\Sigma(?)$	$\Xi(?)$	

Spontaneous χSB

What was wrong with the previous argument?
We have tacitly assumed that the ground state of QCD (vacuum) is annihilated by Q_A^a

To show this, consider a creation operator asociated with positive parity fields $a_i^\dagger$ creating positive parity state $|i,+\rangle$ and $b_j^\dagger$ creates quanta of opposite parity. States $|i,+\rangle$ and $|j,-\rangle$ are basis states of an irreducible representation of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$

In analogy with (lecture 9) $[Q^a(t), \Phi_k(\vec{y},t)] = -t^a_{kj}\Phi_j(\vec{y},t)$

we have

$$[Q_A^a, a_i^\dagger] = -t^a_{ij} b_j^\dagger$$

Then

$$Q_A^a|i,+\rangle = Q_A^a a_i^\dagger|0\rangle = \Big([Q_A^a, a_i^\dagger] + a_i^\dagger \underbrace{Q_A^a}_{\hookrightarrow 0}\Big)|0\rangle = -t^a_{ij} b_j^\dagger|0\rangle$$

If axial charges annihilate vacuum then we arrive at

$$|\phi\rangle = Q_A^a|i,+\rangle = -t^a_{ij}|j,-\rangle$$

What happens when $Q_A^a|0\rangle \neq 0$?

Spontaneous χSB

Goldstone theorem:
For each charge (generator) of some symmetry group that does not annihilate vacuum there corresponds a massless particle (Goldstone boson) of parity equal to the parity of this charge. In QCD natural candidates for Goldstone bosons are: π, K and η.

In QCD $Q_V^a|0\rangle = Q_V|0\rangle = 0$ so the vacuum is invariant under $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$
It follows that H^0_{QCD} is also invariant (but not vice versa) and that the physical states correspond to some irreducible representations of $\mathrm{SU}(3)_V \times \mathrm{U}(1)_V$

To each $Q_A^a|0\rangle \neq 0$ there corresponds a massless Goldstone boson field $\phi^a(x)$ with zero spin and

$$\phi^a(\vec{x},t) \stackrel{P}{\mapsto} -\phi^a(-\vec{x},t)$$

Moreover:

$$[Q_V^a, \phi^b(x)] = i f_{abc}\phi^c(x)$$

Quark masses break axial symmetry explicitly, so Goldstone bosons are not exactly massless.

Quark condensate

Recall definitions

$$\begin{aligned} S_a(y) &= \bar{q}(y)\lambda_a q(y), \quad a = 0, \cdots, 8, \\ P_a(y) &= i\bar{q}(y)\gamma_5\lambda_a q(y), \quad a = 0, \cdots, 8. \end{aligned}$$

Generic quark billinears

$$A_i(x) = q^\dagger(x)\hat{A}_i q(x)$$

have the following commutation rules

$$[A_1(\vec{x}, t), A_2(\vec{y}, t)] = \delta^3(\vec{x} - \vec{y})q^\dagger(x)[\hat{A}_1, \hat{A}_2]q(x)$$

Calculate commutators of vector currents $Q_V^a(t) = \int d^3x q^\dagger(\vec{x}, t)\frac{\lambda^a}{2}q(\vec{x}, t)$ with S and P

we have $[\frac{\lambda_a}{2}, \gamma_0\lambda_0] = 0$ and $[\frac{\lambda_a}{2}, \gamma_0\lambda_b] = \gamma_0 i f_{abc}\lambda_c$

scalar quark densities transform as a singlet and an octet (similarly pseudoscalars)

$$\begin{aligned} [Q_V^a(t), S_0(y)] &= 0, \quad a = 1, \cdots, 8, \\ [Q_V^a(t), S_b(y)] &= i\sum_{c=1}^{8} f_{abc}S_c(y), \quad a, b = 1, \cdots, 8 \end{aligned}$$

Quark condensate

$$[Q_V^a(t), S_b(y)] \;=\; i\sum_{c=1}^{8} f_{abc}S_c(y), \quad a, b = 1, \cdots, 8.$$

One can invert this relation with the help of (recall computation of the Casimir)

$$\sum_{a,b=1}^{8} f_{abc}f_{abd} = 3\delta_{cd}$$

$$S_a(y) = -\frac{i}{3}\sum_{b,c=1}^{8} f_{abc}[Q_V^b(t), S_c(y)]$$

Because vector charges annihilate vacuum $Q_V^a|0\rangle = 0$ we have

$$\langle 0|S_a(y)|0\rangle = \langle 0|S_a(0)|0\rangle \equiv \langle S_a\rangle = 0, \quad a = 1, \cdots, 8$$

where we have used translation invariance of the ground sate:

$$e^{ipy}S(y)e^{-ipy} = S(0)$$

Quark Condensate

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

From $\langle S_a \rangle = 0$ we have:

$a = 3 \quad \langle \bar{u}u \rangle - \langle \bar{d}d \rangle = 0 \qquad\qquad a = 8 \quad \langle \bar{u}u \rangle + \langle \bar{d}d \rangle - 2\langle \bar{s}s \rangle = 0$

From these eqs. we have

$$\langle \bar{u}u \rangle = \langle \bar{d}d \rangle = \langle \bar{s}s \rangle$$

Because $[Q_V^a(t), S_0(y)] = 0, \quad a = 1, \cdots, 8$ the same argument cannot be used for singlet condensate.

However it is clear that

$$0 \neq \langle \bar{q}q \rangle = \langle \bar{u}u + \bar{d}d + \bar{s}s \rangle = 3\langle \bar{u}u \rangle = 3\langle \bar{d}d \rangle = 3\langle \bar{s}s \rangle$$

Quark condensate

Now we shall calculate commutator $i[Q_a^A(t), P_a(y)]$ for fixed a

This is related to

$$(i)^2[\gamma_5 \frac{\lambda_a}{2}, \gamma_0\gamma_5\lambda_a] = \lambda_a^2\gamma_0$$

We have

$$\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4^2 = \lambda_5^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\lambda_6^2 = \lambda_7^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\lambda_8^2 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Quark condensate

Now we shall calculate commutator $i[Q_a^A(t), P_a(y)]$ for fixed a

This is related to

$$(i)^2[\gamma_5 \frac{\lambda_a}{2}, \gamma_0\gamma_5\lambda_a] = \lambda_a^2\gamma_0$$

We have (suppressing y dependence)

$$i[Q_a^A(t), P_a(y)] = \begin{cases} \bar{u}u + \bar{d}d, & a = 1,2,3 \\ \bar{u}u + \bar{s}s, & a = 4,5 \\ \bar{d}d + \bar{s}s, & a = 6,7 \\ \frac{1}{3}(\bar{u}u + \bar{d}d + 4\bar{s}s), & a = 8 \end{cases}$$

which gives vacuum expectation value

$$\langle 0|i[Q_a^A(t), P_a(y)]|0\rangle = \frac{2}{3}\langle \bar{q}q\rangle, \quad a = 1, \cdots, 8$$

Goldstone bosons

Expectation value is non-zero and time independent

$$\langle 0| i[Q_a^A(t), P_a] |0\rangle = i \int d^3x \langle 0| [A_a^0(x), P_a] |0\rangle$$

$$= i \int d^3x \sum\!\!\!\!\!\!\int_n \{\langle 0| A_a^0(x) |n\rangle \langle n| P_a |0\rangle - \langle 0| P_a |n\rangle \langle n| A_a^0(x) |0\rangle\}$$

where
$$\sum\!\!\!\!\!\!\int_n = \sum_n \int \frac{d^4p_n}{(2\pi)^3}\delta(p_n^2 - m_n^2) = \sum_n \int \frac{d^3p_n}{(2\pi)^3 2p_n^0}$$

$$= i \int d^3x \sum\!\!\!\!\!\!\int_n \{e^{-ip_n x} \langle 0| A_a^0(0) |n\rangle \langle n| P_a |0\rangle - e^{ip_n x} \langle 0| P_a |n\rangle \langle n| A_a^0(0) |0\rangle\}$$

only states with zero energy conribute (time indep.)

$$e^{-ip_n x} = e^{-i(p_n^0 t - \boldsymbol{p}_n \boldsymbol{x})}$$

$$\langle 0| i[Q_a^A(t), P_a] |0\rangle = \frac{i}{2} \lim_{p^0 \to 0} \sum_b \int \frac{d^3p}{(2\pi)^3} \int d^3x \left\{ e^{i\boldsymbol{px}} \frac{\langle 0| A_a^0 |\phi^b\rangle}{p^0} \langle \phi^b| P_a |0\rangle - \text{h.c.} \right\}$$

Integral over d^3x gives Dirac delta, which eats up integration over d^3p

Goldstone bosons

$$\langle 0| i[Q_a^A(t), P_a] |0\rangle \;=\; \frac{i}{2}\lim_{p^0\to 0}\sum_b \left\{ \frac{\langle 0| A_a^0 |\phi^b\rangle}{p^0} \langle \phi^b| P_a |0\rangle - \langle 0| P_a |\phi^b\rangle \frac{\langle \phi^b| A_a^0 |0\rangle}{p^0} \right\}$$

From hermicity and Lorentz invariance $\langle 0| A_a^\mu |\phi^b(p)\rangle = ip^\mu F_\phi \delta^{ab}$

and we get

$$\langle 0| i[Q_a^A(t), P_a] |0\rangle = -F_\phi \langle \phi^a| P_a |0\rangle = \frac{2}{3}\langle \bar{q}q\rangle$$

Here F_ϕ is Goldstone boson (pion) decay constant. Its value is ~ 93 MeV (different normalizations).

- There must exist states for which $\langle 0| A_a^0(0) |n\rangle$ and $\langle 0| P_a |n\rangle$ are non-zero
- It is not vacuum, because $\langle 0| P_a |0\rangle = 0$
- Energy of these states must vanish, because the quark condendate is time independent
- So we need $E_n = 0$
- Such states are massless Goldstone bosons $|\phi^b\rangle$
- GBs are (pseudo)scalars – still to be proven