Recall, that for free scalar theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2$$

we define the generating functional

$$Z_0[J] = \frac{\langle 0|\,0\rangle|_J}{\langle 0|\,0\rangle|_{J=0}}$$

and employ the functional integral form

$$\begin{aligned}
\langle 0|\,0\rangle|_J &= \int [\mathcal{D}\varphi(x)] \exp\left\{ i \int d^4x \left(\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2 + J\varphi \right) \right\} \\
&= \int [\mathcal{D}\varphi(x)] \exp\left\{ -i \int d^4x\, \varphi(x) \underbrace{\left(\frac{1}{2}\partial_x^2 + \frac{1}{2}m^2 \right)}_{\frac{1}{2}D_x} \varphi(x) + J(x)\varphi(x) \right\}.
\end{aligned}$$

Define Green's function

$$G(x,y) = \langle 0|\, T\left(\hat{\varphi}(x)\hat{\varphi}(y)\right) |0\rangle = \frac{1}{i^2}\left(\frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} Z_0[J] \right)\Bigg|_{J=0}$$

This implies Klein-Gordon eq. for φ

$$\left(\partial_x^2 + m^2\right)\varphi_c = J$$

We can solve this by means of the Klein-Gordon Green's function

$$\left(\partial_x^2 + m^2\right)\Delta(x,y) = -i\delta^{(4)}(x-y).$$

Then

$$\varphi(x) = i\int d^4y\, \Delta(x,y) J(y).$$

Indeed

$$\left(\partial_x^2 + m^2\right)\varphi(x) = i\int d^4y\, \left(\partial_x^2 + m^2\right)\Delta(x,y)J(y) = J(x).$$

Now we can do Fourier transform

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \tilde{\Delta}(k)$$

and

$$\begin{aligned}
\int \frac{d^4k}{(2\pi)^4} \left(\partial_x^2 + m^2\right) e^{ik(x-y)} \tilde{\Delta}(k) &= -i\int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \\
\int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \left(-k^2 + m^2\right) \tilde{\Delta}(k) &= -i\int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)}.
\end{aligned}$$

We have

$$\left(k^2 - m^2\right) = i.$$

We can write

$$\tilde{\Delta}(k) = \frac{i}{k^2 - m^2}$$

only if the inverse exists. Here we only need $i\varepsilon$ prescription, but in gauge theories we need gauge fixing.

Let's write

$$\langle 0 | \, 0 \rangle |_J = \int [\mathcal{D}\varphi(x)] \exp\left\{ -i\frac{1}{2} \int d^4x \, \varphi(x) \left(\partial_x^2 + m^2\right) \varphi(x) + i \int d^4x \, J(x)\varphi(x) \right\}$$

and then use Fourier transform

$$\begin{aligned} & \int d^4x \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} e^{ilx} \tilde{\varphi}(l) \left(\partial_x^2 + m^2\right) e^{ikx} \tilde{\varphi}(k) \\ = & \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4l}{(2\pi)^4} \int d^4x e^{i(k+l)x} \tilde{\varphi}(l)(-k^2 + m^2)\tilde{\varphi}(k) \\ = & \int \frac{d^4k}{(2\pi)^4} \tilde{\varphi}(-k)(-k^2 + m^2)\tilde{\varphi}(k). \end{aligned}$$

We immediately see what opertor we have to invert, but in order to perform Gaussian integral over $[\mathcal{D}\varphi(x)]$ we need to do Fourier transform once more:

$$\int \frac{d^4k}{(2\pi)^4} \tilde{\varphi}(-k)(-k^2 + m^2)\tilde{\varphi}(k) = \int d^4x \int d^4y \, \varphi(y) \underbrace{\int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} (-k^2 + m^2)}_{\sim \Delta^{-1}(x-y)} \varphi(x).$$

We never need integration over $[\mathcal{D}\tilde{\varphi}(k)]$.