QCD
problem set 12

1. For (iso)spin J lowering and raising operators act on the basis states in the following way:
$$J_\pm \,|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j,m\pm 1\rangle \tag{1}$$
States $|1,m\rangle$ are given by linear combinations of basis states $|\phi_i\rangle$ ($i = 1,2,3$) in the *natural* basis where
$$\tilde{J}^i_{jk} = -i\varepsilon_{ijk}.$$
These combinantions and explicit form of matrices $\tilde{J}^i_{jk}$were given at lecture 12. Convince yourself by explicit calculations that operators $\tilde{J}^\pm_{jk}$ defined in natural basis fulfill (1) if the phases of states $|j,m\rangle$ are chosen as at the lecture.

2. At lecture 11 there was a misprint in the definition of a scalar field decomposition in terms of annihilation and creation operators (factor $(2\pi)^3$ instead $(2\pi)^{3/2}$). Canonical commutation relation
$$\left[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{x}\,')\right] = i\delta^{(3)}(\vec{x}-\vec{x}\,')$$
where $\hat{\pi}(t,\vec{x}) = \partial_t\hat{\phi}(t,\vec{x})$ (why?) implies certain commutation rule for $\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')\right]$ where
$$\hat{\phi}(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}\sqrt{2\omega_k}}\left[e^{-i\,kx}\hat{a}(\vec{k}) + e^{+i\,kx}\hat{a}^\dagger(\vec{k})\right].$$
Find this commutation rule.

3. For on-shell heavy quark of mass M_Q one can write that its momentum is given as
$$p^\mu = M_Q v^\mu$$
provided $v^2 = 1$. Show that operators
$$\frac{1}{2}\left(1 \pm \not{v}\right)$$
satisfy all necessary conditions of projection operators.

4. Pove that
$$\frac{1+\not{v}}{2}\gamma^\mu\frac{1+\not{v}}{2} = \frac{1+\not{v}}{2}v^\mu\frac{1+\not{v}}{2}$$

5. Solutions of the Dirac equation in the Dirac (Bjorken-Drell) representation of γ matrices take the following form:

$$u(p,s) = \sqrt{E_p + m} \begin{bmatrix} \chi(s) \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m}\chi(s) \end{bmatrix}, \quad v(p,s) = \sqrt{E_p + m} \begin{bmatrix} \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E_p+m}\chi(s) \\ \chi(s) \end{bmatrix}, \tag{2}$$

where $s = 1, 2$ labels spin. Solutions denoted by u correspond to particles, while solutions denoted by v to antiparticles (E_p is defined to be positive). Calculate action of the projection operators defined in the previous problem for a heavy fermion at rest.