QCD
problem set 11

1. Flavor SU(3) left and right currents read

$$L^a_\mu = \bar{q}_L \gamma_\mu \frac{\lambda_a}{2} q_L, \; R^a_\mu = \bar{q}_R \gamma_\mu \frac{\lambda_a}{2} q_R$$

where

$$q_{\substack{L \\ R}} = \frac{1}{2}\left(1 \mp \gamma_5\right) q.$$

Show that vector and axial currents take the following form

$$\begin{aligned} V^a_\mu &= R^a_\mu + L^a_\mu = \bar{q}\gamma_\mu \frac{\lambda_a}{2} q, \\ A^a_\mu &= R^a_\mu - L^a_\mu = \bar{q}\gamma_\mu \gamma_5 \frac{\lambda_a}{2} q. \end{aligned}$$

2. In the case of fermion fields, commutation relations of scalar fields are replaced by anticommutation relations:

$$\left\{ q_{\alpha,k}(t,\vec{x}), q^\dagger_{\beta,l}(t,\vec{x}\,') \right\} = \delta^{(3)}(\vec{x}-\vec{x}\,')\delta_{\alpha\beta}\delta_{kl}$$

where α, β stand for Dirac indices and k, l denote SU(3) indices. Relevant charges are defined as

$$\begin{aligned} \hat{Q}^a_{L,R}(t) &= \int d^3\vec{x}\, q^\dagger_{L,R}(t,\vec{x}) T^a q_{L,R}(t,\vec{x}), \\ \hat{Q}_V(t) &= \int d^3\vec{x} \left[q^\dagger_L(t,\vec{x}) q_L(t,\vec{x}) + q^\dagger_R(t,\vec{x}) q_R(t,\vec{x}) \right] \end{aligned}$$

where $T^a = \lambda^a/a$ are SU(3) generators (Gell-Mann matrices). Making use of the identity (prove it!)

$$\{ab, cd\} = a\{b,c\}d - ac\{b,d\} + \{a,c\}bd - c\{a,d\}b$$

show that

$$\begin{aligned} \left[\hat{Q}^a_L, \hat{Q}^b_L\right] &= i f^{abc} \hat{Q}^c_L, \\ \left[\hat{Q}^a_R, \hat{Q}^b_R\right] &= i f^{abc} \hat{Q}^c_R, \\ \left[\hat{Q}^a_L, \hat{Q}^b_R\right] &= 0, \\ \left[\hat{Q}^a_{L,R}, \hat{Q}_V\right] &= 0. \end{aligned}$$

3. Effective Lagrangian describing Goldstone boson interactions reads

$$\mathcal{L}_{\rm eff} = \frac{F^2}{4}\,\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)$$

where $U = \exp(i\phi/F)$ can be expressed in terms of the physical meson fields:

$$\phi(x) = \sum_a \lambda_a \phi^a(x) = \begin{pmatrix} \pi^0 + \eta/\sqrt{3} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \eta/\sqrt{3} & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -2\eta/\sqrt{3} \end{pmatrix}$$

Expand $\mathcal{L}_{\rm eff}$ up to 4-field interactions and calculate flavor trace for the case of SU(2) (see lecture 11) and SU(3).

4. Decay width Γ (inverse of the particle life-time) of a particle of mass M (at rest) into two particles of masses $m_{1,2}$ reads

$$\Gamma = \frac{1}{2M}\int |\mathcal{M}|^2\,(2\pi)^4\,\delta^{(4)}(p-k_1-k_2)\,\frac{d^4k_1}{(2\pi)^4}\,(2\pi)\,\delta(k_1^2-m_1^2)\frac{d^4k_2}{(2\pi)^4}\,(2\pi)\,\delta(k_2^2-m_2^2)$$

where $\mathcal{M}$ is the relevant matrix element. Show that one can perform all integrations in the above formula. As a first step perform integrations over $dE_{1,2}$. Then using $\delta^{(4)}(p-k_1-k_2) = \delta(M-E_1-M_2)\delta^{(3)}(\boldsymbol{k}_1+\boldsymbol{k}_2)$ perform integration over $d^3\boldsymbol{k}_2$.Using

$$d^3\boldsymbol{k}_1 = k_1^2 dk_1 d\Omega$$

and assuming that matrix element $\mathcal{M}$ does not depend on angles perform integration over $d\Omega$. Finally, using the energy δ function perform the integration over $dk_1$with the help of the following change of variables:

$$u = \sqrt{k_1^2+m_1^2} + \sqrt{k_1^2+m_2^2}.$$

Solve the energy δ function assuming that particle 2 is massless.