QCD
problem set 8

1. Winding number of the SU(2) gauge transformation U is defined as
$$N_{\mathrm{w}} = \frac{1}{24\pi^2}\varepsilon^{ijk}\int d^3r \; \mathrm{Tr}\left[\left(U^\dagger\partial_i U\right)\left(U^\dagger\partial_j U\right)\left(U^\dagger\partial_k U\right)\right]. \tag{1}$$
Calculate (1) for $U = \exp\left(i\,\vec{n}\cdot\vec{\tau}\,P(r)\right)$ where $\vec{n} = \vec{r}/r$. What are the boundary conditions for $P(r)$ that ensure that N_{w} is an integer?

   HINT:

   First decompose
$$U^\dagger\partial_i U = \frac{i}{2}\sum_{a=1}^{3}\xi_i^a\tau_a$$
where τ_a are Pauli matrices. You should obtain that $\varepsilon^{ijk}\,\mathrm{Tr}\left[\left(U^\dagger\partial_i U\right)\left(U^\dagger\partial_j U\right)\left(U^\dagger\partial_k U\right)\right] \sim \det(\xi)$.

   Due to the symmetry of U, elements of matrix ξ can be decomposed in the following way:
$$\xi_i^a = A\delta_{ia} + Bn_in_a + C\varepsilon_{iak}n_k.$$
Express $\det(\xi)$ in terms of A, B and C.You should get an answer, which is proportional to $(A^2+C^2)(A+B)$.

   In the last step calculate A, B and C. To this end expand U using de'Moivre (or Euler) formula for the exponent. For this you have to prove that $(\vec{n}\cdot\vec{\tau})^2 = 1$.

   In order to differentiate U it is useful to use the following identities (prove them!)
$$\begin{aligned}\partial_i r &= n_i,\\ \partial_i r_k &= \frac{1}{r}(\delta_{ik} - n_in_k).\end{aligned}$$

2. Calculade the field tensor $F_{\mu\nu}$ for the pure gauge $A_\mu = -\frac{i}{g}U\partial_\mu U^\dagger$.

3. Prove that
$$\int\limits_{-T/2}^{T/2} d\tau_1 \int\limits_{\tau_1}^{T/2} d\tau_2 \ldots \int\limits_{\tau_{n-1}}^{T/2} d\tau_n = \frac{1}{n!}T^n.$$