QCD
problem set 2

Problem classes will take place on Tuesdays (starting on October 20) at 1:30 pm in the lecture hall D-2-02 (theory seminar room). Lectures are shifted to Wednesdays at 3:15 pm, room A-1-04.

The purpose of the present set is to calculate the cross-section for elastic scattering of two different elementary fermions. We start with the textbook formula for the $p_1 + p_2 \to q_1 + q_2 + \dots q_n$ scattering (here p_i denote four-momenta of two incoming particles having masses m_i, and q_j momenta of n outgoing particles having masses M_j):

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{1}{4} \int \sum_{\text{pol}} |\mathcal{M}_{fi}|^2$$
$$\left[\prod_{j=1}^{n} \frac{d^4 q_j}{(2\pi)^3} \delta_+(q_j^2 - M_j^2) \right] (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2 - \dots q_n).$$

The first factor with the square root is called the flux factor and it measures the probability that the incoming particles had any interaction at all. Because we do not measure spins, neither are the incoming fermions polarized, we have the sum over spin polarizations of incoming and outgoing particles. Factor $1/4$ takes care of the fact that we average over the spin projections of the incoming particles. $\mathcal{M}_{fi}$ is the scattering amplitude to be calculated according to the Feynman rules for QED. For each outgoing particle we integrate over its four momentum with the constraint that it is on the mass-shell (the $+$ sign in the Dirac delta implies that only positive energy contributes to the expression for the cross-section). The last Dirac delta takes care of the energy momentum conservation. Although we have put explicitly the integral symbol, not all integrals have to be performed. We do not perform the integrals over the variables that we measure. Some integrals are easily performed with the help of the Dirac deltas.

Consider elastic scattering of an electron of mass $m = 0$ on a proton of mass M, assuming that proton is elementary. In this case outgoing particles are the same as the incoming ones and they differ only by final momenta. Assume that before the scattering the proton is at rest and the electron moves along the z axis

$$\begin{aligned} p_1 &= k = \omega(1, 0, 0, 1), \\ p_2 &= p = M(1, 0, 0, 0). \end{aligned}$$

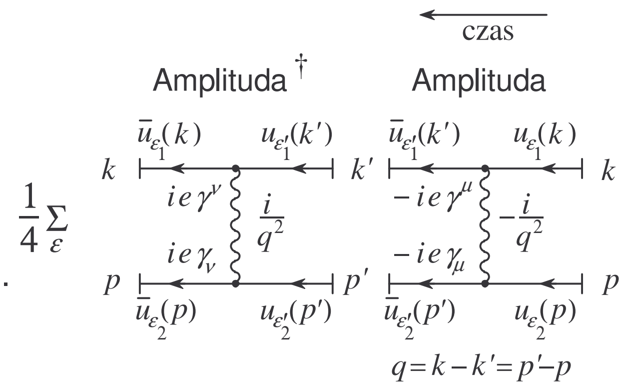
After the scattering electron four-momentum takes the following form

$$q_1 = k' = \omega'(1, \sin\theta \sin\varphi, \sin\theta \cos\varphi, \cos\theta).$$

We do not measure the proton four-mometum $q_2 = p'$ but it is completely determined by the energy momentum conservation. Energy transfer is denoted by $\nu = \omega - \omega'$ and the four-momentum transfer is denoted by $q = k - k'$.

1. Calculate the flux factor for the above kinematics.
2. Perform the integral over $d^4q_2 = d^4p'$ with the help of the last Dirac delta.
3. Express $Q^2 = -q^2$ in terms of ω, ω' and $\cos\theta$.
4. Express the integral over d^4k' in terms of $d\omega' d\cos\theta\, d\varphi$. Assuming that $\mathcal{M}_{fi}$ does not depend on φ, perform the integral over $d\varphi$. The remaining integral over $d\omega' d\cos\theta$ can be expressed in terms of $dQ^2 d\nu$. Calculate the Jacobian for this change of variables.
5. Express the on-shell condition for the proton in terms of Q^2 and ν.

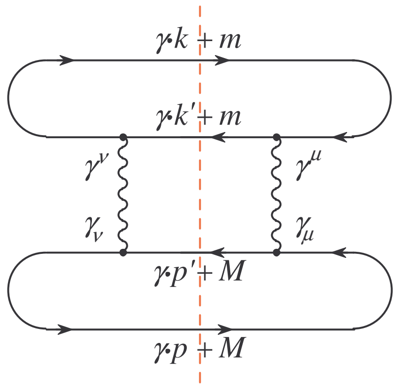
   We cannot proceed further without calculating $|\mathcal{M}_{fi}|^2$. Feynman rules for QED can be found in many textbooks, see *e.g.* Bjorken, Drell, *Relativistic Quantum Mechanics,* or Greiner *QED*. Amplitude and Hermitean conjugate of the amplitude for elastic scattering, together with all relevant factors are shown in figure below (note that time (*czas*) is flowing from right to left). Here u and $\bar{u}$ denote Dirac spinor (and its Dirac conjugate) for initial and final particles.



6. With the help of the following identity for the sum over spin polarizations:

$$\sum_{\varepsilon} [u_\varepsilon(p)]_\alpha \, [\bar{u}_\varepsilon(p)]_\beta = (\not{p} + m)_{\alpha\beta}$$

   where $\alpha, \beta = 1, \dots 4$ are Dirac spinor indices and $\not{p} = \gamma^\mu p_\mu$ is so called Dirac *slash* (where γ^μ denote Dirac matrices), show that the square of the amplitude $\sum_{\text{pol}} |\mathcal{M}_{fi}|^2$ can be reduced to the contraction of two tensors, each of them being a trace of Dirac matrices (or unit matrix) represented by the figure below:



and is given by the formula

$$\frac{1}{4}\sum_{\text{pol}} |\mathcal{M}_{fi}|^2 = \frac{e_1^2 e_2^2}{(q^2)^2} L^{\nu\mu}(k,k') L_{\nu\mu}(p,p').$$

7. Dirac traces can be calculated by techniques described in the textbooks, you can also calculate them using Wolfram *Mathematica*. Express each of them in terms of k^μ and q^μ, or p_μ and q_μ. Check that they are gauge invariant, *i.e.*

$$q_\nu L^{\nu\mu} = q_\mu L^{\nu\mu} = 0.$$

Try to decompose $L_{\nu\mu}$ into a sum of two pieces, each being seaparetely gauge invariant.

8. Show that

$$\begin{aligned} p_\nu p_\mu L^{\nu\mu}(k,k') &= 4M^2 \omega\omega' \cos^2\frac{\theta}{2}, \\ g_{\nu\mu} L^{\nu\mu}(k,k') &= -8\omega\omega' \sin^2\frac{\theta}{2}. \end{aligned}$$

9. Calulate final expression for

$$\frac{d\sigma}{dQ^2}.$$